A MAXIMIZATION PROBLEM IN THE THEORY OF PARTITIONS

by

Russell A. Smucker

TECHNICAL REPORT NO. 25

PREPARED UNDER RESEARCH GRANT NO. NsG-568
(PRINCIPAL INVESTIGATOR: T. N. BHARGAVA)

FOR

NATIONAL AERONAUTICS and SPACE ADMINISTRATION

DEPARTMENT OF MATHEMATICS

KENT STATE UNIVERSITY

KENT, OHIO

June 1967

SMUCKER, RUSSELL A., M.A., September, 1967 MATHEMATICS

A MAXIMIZATION PROBLEM IN THE THEORY OF PARTITIONS (48 pp.)

Director of Thesis: T. N. Bhargava

The purpose of this thesis is to study two generalizations of the following problem in the theory of partitions of integers: maximize the sum of the products of all pairs of adjacent terms in an ordered p-tuple subject to the restriction that the coordinates be positive integers whose sum is fixed ($p \geq 2$).

In chapter I we consider the problem of maximizing the sum of the products of all ordered q-tuples of adjacent terms in an ordered p-tuple subject to the restriction that the coordinates be positive integers whose sum is fixed ($2 \leq q \leq p$). We give an explicit description of a p-tuple which yields the maximum and use it to find an approximation for the maximum.

In chapter II we alter the problem by allowing the coordinates to be nonnegative real numbers. We again display a p-tuple yielding the maximum, and we use it to find an explicit expression for the maximum.

SMUCKER, RUSSELL A., M.A., September, 1967 MATHEMATICS

A MAXIMIZATION PROBLEM IN THE THEORY OF PARTITIONS (48 pp.)

Director of Thesis: T. N. Bhargava

The purpose of this thesis is to study two generalizations of the following problem in the theory of partitions of integers: maximize the sum of the products of all pairs of adjacent terms in an ordered p-tuple subject to the restriction that the coordinates be positive integers whose sum is fixed ($p \geq 2$).

In chapter I we consider the problem of maximizing the sum of the products of all ordered q-tuples of adjacent terms in an ordered p-tuple subject to the restriction that the coordinates be positive integers whose sum is fixed ($2 \leq q \leq p$). We give an explicit description of a p-tuple which yields the maximum and use it to find an approximation for the maximum.

In chapter II we alter the problem by allowing the coordinates to be nonnegative real numbers. We again display a p-tuple yielding the maximum, and we use it to find an explicit expression for the maximum.

A MAXIMIZATION PROBLEM IN
THE THEORY OF PARTITIONS

A thesis submitted to the
Kent State University Graduate School
in partial fulfillment of the requirements
for the degree of Master of Arts

by

Russell Smucker

June 1967

Acknowledgments

I am grateful to Dr. T. N. Bhargava for his encouragement and guidance throughout the development of this thesis.

For financial support, I am indebted to the National Aeronautics and Space Administration through NASA Research Grant Number NsG-568.

Table of Contents

Chapter 0

Preliminaries

The purpose of this thesis is to study two generalizations of a problem in the theory of partitions of integers: maximize the sum of products of pairs of adjacent terms in an ordered p-tuple subject to the restriction that the coordinates be positive integers whose sum is fixed ($2\leq p$).

We are thankful to Professor S. D. Chatterji for bringing this problem to our attention. The generalizations we consider are: (a) solve the above problem, but take the sum of q+1-tuples of adjacent terms ($1\leq q\leq p-1$); (b) solve (a) allowing the coordinates to be nonnegative real numbers.

We state some definitions and notations.

Definition Let N and p be positive integers such that $2\leq p\leq N$. An ordered p-tuple of positive integers the sum of whose coordinates is N is said to be an ordered partition of N having length p.

The notation $\Pi(N,p)$ is used to denote the set of all ordered partitions of N having length p. Throughout this thesis N and p are considered fixed; so N and p will be suppressed. Therefore $\Pi(N,p)$ will be written simply as Π.

Let $\pi=(x_1,\ldots,x_p)\epsilon\Pi$, and let q be a positive integer such that $1\leq q\leq p-1$. The set of positive integers is denoted by I^+. The mapping $\sigma_q : \Pi \longrightarrow I^+$ is defined by

$$\sigma_q(\pi) = \sum_{i=1}^{p-q} x_i x_{i+1} \cdots x_{i+q}.$$

The integer q is considered fixed and as such will be suppressed. So σ_q will be written as σ.

The number μ is defined by $\mu = \max_{\pi\varepsilon\Pi} \sigma(\pi)$. We immediately notice that μ exists, because Π is nonempty and finite.

The first problem we consider in this thesis is to express μ in terms of the three fixed parameters N, p, and q, and to display a partition $\bar{\pi}\varepsilon\Pi$ such that $\mu = \sigma(\bar{\pi})$. This problem is treated in chapter I.

In chapter II we solve a certain generalization of the problem to real numbers.

<u>Definition</u> Let α be a positive real number, and let p be a positive integer such that $2 \leq p$. An ordered p-tuple of nonnegative real numbers the sum of whose coordinates is α is said to be an <u>ordered real partition of α having length p</u>.

$\Pi(\alpha,p)$, σ_q, and μ are defined for the real numbers as the obvious analogues of the definitions for the integers. Again, α, p, and q are considered fixed and will be suppressed in notation. In chapter II we prove the existence of μ, find an expression for μ in terms of α, p, and q, and display a partition $\bar{\pi}\varepsilon\Pi$ such that $\mu = \sigma(\bar{\pi})$.

Chapters I and II are self-contained; so the suppression of both N and α in notation does not lead to ambiguity in the meaning of Π as $\Pi(N,p)$ or $\Pi(\alpha,p)$.

To facilitate using a simpler notation we state two conventions. $\sum_{i=a}^{b} f(i)=0$ if $a>b$. If $i<1$ or $i>p$, then $x_i=0$.

Brackets are used exclusively to denote the greatest integer function. The notation $\bar{\pi}=(\bar{x}_1,\ldots,\bar{x}_p)$ is reserved for $\bar{\pi}$ such that $\mu=\sigma(\bar{\pi})$, i.e. a bar indicates a partition which yields the maximum μ.

Given an arbitrary partition $\pi=(x_1,\ldots,x_p)$ in Π (either $\Pi(N,p)$ or $\Pi(\alpha,p)$), we construct new partitions in Π as follows:

$\hat{\pi} = (y_1,\ldots,y_p)$, defined by

$y_i=x_{p-i+1}$ for $i=1,\ldots,p$;

$\overrightarrow{\pi} = (y_1,\ldots,y_p)$, defined by

$y_1=x_p$,

$y_i=x_{i-1}$ for $i=2,\ldots,p$; and

$\overleftarrow{\pi} = (y_1,\ldots,y_p)$, defined by

$y_p=x_1$,

$y_i=x_{i+1}$ for $i=1,\ldots,p-1$.

Let h and k be distinct fixed integers, $1\le h,k\le p$. If $\pi=(x_1,\ldots,x_p)\epsilon\Pi(N,p)$ is such that $x_k\ge 2$, then $\pi\langle h,k\rangle=(y_1,\ldots,y_k)\epsilon\Pi(N,p)$ is defined by

$y_h = x_h + 1$,

$y_k = x_k - 1$,

$y_i = x_i$ for $i \neq h,k$.

In the real case, if $\pi = (x_1, \ldots, x_p) \varepsilon \Pi(\alpha, p)$ is such that $x_k \geq \varepsilon > 0$, (ε a real number), then $\pi\langle h,k\rangle = (y_1, \ldots, y_p) \varepsilon \Pi(\alpha, p)$ is defined by

$y_h = x_h + \varepsilon$

$y_k = x_k - \varepsilon$

$y_i = x_i$ for $i \neq h,k$.

In the real case $\pi\langle h,k\rangle$ depends on ε; so ε must be explicitly stated for $\pi\langle h,k\rangle$ to make sense.

Chapter I

Maximizing Over Positive Integers

We find in this chapter a partition $\bar{\pi}$ such that $\mu=\sigma(\bar{\pi})$. Using $\bar{\pi}$ we obtain good, simple bounds for μ. Our variables x_i are positive integers. The case $N=p$ is trivial; so it is assumed henceforth that $N>p$.

1.1 The Case $q=1$

In this section we consider the simplest case, viz. $q=1$. The results of the next section imply the results of this section. However, we present this case separately to illustrate in simple form some of the techniques used in the more general setting. The results given in this section are essentially those of Professor A. Meir, University of Alberta, and they were communicated to us in a letter. For this we are thankful to Professor Meir.

Theorem 1.1. For $q=1$ the maximum μ is given by:

For $p=2,3$, $\quad \mu=\left[\frac{N}{2}\right]\left[\frac{N+1}{2}\right]$.

For $p>3$, $\quad \mu=2N-p-1+\left[\frac{N-p}{2}\right]\left[\frac{N-p+1}{2}\right]$.

A partition $\bar{\pi}$ is given by:

For $p=2$, $\quad \bar{\pi}=\left(\left[\frac{N}{2}\right],\left[\frac{N+1}{2}\right]\right)$.

For $p=3$, $\quad \bar{\pi}=\left(1,\left[\frac{N}{2}\right],\left[\frac{N-1}{2}\right]\right)$.

For $p>3$, $\quad \bar{\pi}=\left(\underbrace{1,1,\ldots,1}_{p-3},\left[\frac{N-p+2}{2}\right],\left[\frac{N-p+3}{2}\right],1\right)$.

Proof. The proof is given in three cases.

Case 1. $p=2$ We prove that in $\bar{\pi}$, $|\bar{x}_1-\bar{x}_2|\leq 1$. Let $\pi\varepsilon\Pi$ be such that $x_1-x_2\geq 2$. Then

$$\begin{aligned}
\sigma(\pi<2,1>) &= y_1y_2 \\
&= (x_1-1)\ (x_2+1) \\
&= x_1x_2+(x_1-x_2-1) \\
&> x_1x_2 \\
&= \sigma(\pi).
\end{aligned}$$

Because $\mu\geq\sigma(\pi<2,1>) > \sigma(\pi)$, it must be true that $\mu\neq\sigma(\pi)$.

Similarly, if $\pi=(x_1,x_2)\varepsilon\Pi$ is such that $x_2-x_1\geq 2$, a symmetric argument shows that $\mu\neq\sigma(\pi)$.

So it must be that in $\bar{\pi}=(\bar{x}_1,\bar{x}_2)$, $|\bar{x}_1-\bar{x}_2|\leq 1$. So $\bar{\pi}$ can be taken to be $\left(\left[\frac{N}{2}\right],\ \left[\frac{N+1}{2}\right]\right)$, from which μ is computed to be $\left[\frac{N}{2}\right]\ \left[\frac{N+1}{2}\right]$.

Case 2. $p=3$ $\sigma(\bar{\pi})= \bar{x}_2(\bar{x}_1+\bar{x}_3)$. If $\bar{x}_2$ and $\bar{x}_1+\bar{x}_3$ are considered as only two variables their sum is still N, and the problem reduces to the case $p=2$. It is sufficient to pick $\bar{x}_2=\left[\frac{N}{2}\right]$ and $\bar{x}_1+\bar{x}_3=\left[\frac{N+1}{2}\right]$. So $\bar{\pi}$ can be chosen as $\bar{\pi}=\left(1,\left[\frac{N}{2}\right],\left[\frac{N-1}{2}\right]\right)$, and $\sigma=\left[\frac{N}{2}\right]\left[\frac{N+1}{2}\right]$. This particular choice for $\bar{\pi}$ is made because it coincides with a more general statement about $\bar{\pi}$ in the next section.

Case 3. $p>3$ First we prove $\bar{x}_1=\bar{x}_p=1$. Let $\pi\varepsilon\Pi$ be such that $x_1\geq 2$. Then

$$\begin{aligned}
\sigma(\pi<3,1>) &= y_1y_2+y_2y_3+y_3y_4+\sum_{i=4}^{p-1} y_iy_{i+1} \\
&= (x_1-1)x_2+x_2(x_3+1)+(x_3+1)x_4+\sum_{i=4}^{p-1} x_ix_{i+1} \\
&= x_1x_2+x_2x_3+x_3x_4+\sum_{i=4}^{p-1} x_ix_{i+1}+x_4
\end{aligned}$$

$= \sigma(\pi)+x_4$

$> \sigma(\pi)$.

So $\mu \neq \sigma(\pi)$.

Similarly if in π $x_p \geq 2$, then $\mu \neq \sigma(\pi)$. This establishes that in $\bar{\pi}$, $\bar{x}_1 = \bar{x}_p = 1$.

We now prove the following by induction on j: for j in the range $1 \leq j \leq p-3$, there exists a partition $\bar{\pi}^* \varepsilon \Pi$ such that $\mu = \sigma(\bar{\pi}^*)$ and $\bar{x}_i^* = 1$ for $i=1,\ldots,j,p$.

For j=1 the existence of such a partition has already been proved, for select any partition $\pi \varepsilon \Pi$ such that $\mu = \sigma(\pi)$.

Now assume that for $1 \leq j \leq p-4$ there is a partition $\bar{\pi}^\circ \varepsilon \Pi$ such that $\mu = \sigma(\bar{\pi}^\circ)$ and $\bar{x}_i^\circ = 1$ for $i=1,\ldots,j,p$. We seek to prove the existence of a partition $\bar{\pi}^* = (\bar{x}_1^*,\ldots,\bar{x}_p^*)$ such that $\mu = \sigma(\bar{\pi}^*)$ and $\bar{x}_i^* = 1$ for $i=1,\ldots,j+1,p$. We use induction on $\bar{x}_{j+1}^\circ$ to show $\bar{\pi}^*$ exists.

If $\bar{x}_{j+1}^\circ = 1$, the existence of π^* is trivial, for let $\bar{\pi}^* = \bar{\pi}^\circ$.

Assume now that if $\bar{x}_{j+1}^\circ = D-1$, the partition π^* exists. Now let $\bar{x}_{j+1}^\circ = D \geq 2$. Consider $\bar{\pi}^\circ \langle j+3, j+1 \rangle$.

$$\sigma(\bar{\pi}^\circ\langle j+3,j+1\rangle) = \sum_{i=1}^{j} y_i y_{i+1} + y_{j+1}y_{j+2} + y_{j+2}y_{j+3} + y_{j+3}y_{j+4} + \sum_{i=j+4}^{p-1} y_i y_{i+1}$$

$$= \sum_{i=1}^{j} \bar{x}_i^\circ \bar{x}_{i+1}^\circ + \bar{x}_{j+2}^\circ + \bar{x}_{j+2}^\circ(\bar{x}_{j+1}^\circ + \bar{x}_{j+3}^\circ - 1) + (\bar{x}_{j+1}^\circ + \bar{x}_{j+3}^\circ - 1)\bar{x}_{j+4}^\circ + \sum_{i=j+4}^{p-1} \bar{x}_i^\circ \bar{x}_{i+1}^\circ$$

$$= \sum_{i=1}^{p-1} \bar{x}_i^\circ \bar{x}_{i+1}^\circ + (\bar{x}_{j+1}^\circ - 1)\bar{x}_{j+4}^\circ$$

$\geq \sigma(\bar{\pi}^{\circ})$

$= \mu$.

Because $y_{j+1} = D-1$, we can invoke the second inductive hypothesis to assert the existence of $\bar{\pi}^*$ for $\bar{\pi}^{\circ}\langle j+3, j+1\rangle$ and hence for $\bar{\pi}^{\circ}$ in which $\bar{x}^{\circ}_{j+1} = D$. This completes the induction.

In particular, for $j=p-3$ there exists a partition $\bar{\pi}$ such that $\bar{x}_i = 1$ for $i=1,\ldots,p-3,p$. We show next that $|\bar{x}_{p-2} - \bar{x}_{p-1}| \leq 1$.

Let π be such that $x_i = 1$ for $i=1,\ldots,p-3,p$, but such that $x_{p-2} - x_{p-1} \geq 2$. Then

$$\begin{aligned}
\sigma(\pi\langle p-1,p-2\rangle) &= \sum_{i=1}^{p-4} y_i y_{i+1} + y_{p-3}y_{p-2} + y_{p-2}y_{p-1} + y_{p-1}y_p \\
&= \sum_{i=1}^{p-4} x_i x_{i+1} + x_{p-3}(x_{p-2}-1) + (x_{p-2}-1)(x_{p-1}+1) \\
&\qquad + (x_{p-1}+1)x_p \\
&= \sum_{i=1}^{p-1} x_i x_{i+1} + x_{p-2} - x_{p-1} - 1 \\
&\geq \sigma(\pi)+1 \\
&> \sigma(\pi).
\end{aligned}$$

So $\mu \neq \sigma(\pi)$.

Similarly if $\pi \in \Pi$ is such that $x_i = 1$ for $i=1,\ldots,p-3,p$ but such that $x_{p-1} - x_{p-2} \geq 2$, then $\mu \neq \sigma(\pi)$.

Consequently, for the choice of $\bar{\pi}$ above it must also be true that $|\bar{x}_{p-2} - \bar{x}_{p-1}| \leq 1$.

So $\bar{\pi}$ can be chosen to be

$$\bar{\pi} = \left(1, 1, \ldots, 1, \left[\frac{N-p+2}{2}\right], \left[\frac{N-p+3}{2}\right], 1\right).$$

From this μ can be computed to be

$$\mu = 2N-p-1 + \left[\frac{N-p}{2}\right]\left[\frac{N-p+1}{2}\right]. \blacksquare$$

1.2 The General Case

In this section we solve the problem of maximizing over the positive integers in the general case. In section 1.1 we obtained our results by comparing $\sigma(\pi)$ and $\sigma(\pi\langle h,k\rangle)$ for some astute choice of h and k. This technique is valuable in the general case as well. So we first state the exact relationships between $\sigma(\pi)$ and $\sigma(\bar{\pi})$, $\sigma(\tilde{\pi})$, and $\sigma(\pi\langle h,k\rangle)$. Evidently $\sigma(\pi)=\sigma(\hat{\pi})$.

Lemma 1.2.1 Let the distinct integers h and k in the range $1\leq h,k\leq p$ be such that $|h-k|\leq q$. Let $\pi=(x_1,\ldots,x_p)\in\Pi$ be such that $x_k\geq 2$. Then

(i) If $k<h$, $\sigma(\pi\langle h,k\rangle)-\sigma(\pi)=$

$$\frac{x_k-x_h-1}{x_h x_k}\sum_{h-q}^{k} x_i\cdots x_{i+q}+\frac{1}{x_h}\sum_{k+1}^{h} x_i\cdots x_{i+q}-\frac{1}{x_k}\sum_{k-q}^{h-q-1} x_i\cdots x_{i+q}\,.$$

(ii) If $h<k$, $\sigma(\pi\langle h,k\rangle)-\sigma(\pi)=$

$$\frac{x_k-x_h-1}{x_h x_k}\sum_{k-q}^{h} x_i\cdots x_{i+q}+\frac{1}{x_h}\sum_{h-q}^{k-q-1} x_i\cdots x_{i+q}-\frac{1}{x_k}\sum_{h+1}^{k} x_i\cdots x_{i+q}\,.$$

Proof of (i). $\sigma(\pi\langle h,k\rangle)$

$$=\sum_{1}^{k-q-1} y_i\cdots y_{i+q}+\sum_{k-q}^{h-q-1} y_i\cdots y_{i+q}$$

$$+\sum_{h-q}^{k} y_i\cdots y_{i+q}+\sum_{k+1}^{h} y_i\cdots y_{i+q}$$

$$+\sum_{h+1}^{p-q} y_i\cdots y_{i+q}$$

$$=\sum_{1}^{k-q-1} x_i\cdots x_{i+q}+\frac{x_k-1}{x_k}\sum_{k-q}^{h-q-1} x_i\cdots x_{i+q}+$$

$$+\frac{(x_k-1)(x_h+1)}{x_h x_k}\sum_{h-q}^{k} x_i\cdots x_{i+q}+\frac{x_h+1}{x_h}\sum_{k+1}^{h} x_i\cdots x_{i+q}$$

$$+\sum_{h+1}^{p-q} x_i\cdots x_{i+q}.$$

$$=\sigma(\pi)+\frac{x_k-x_h-1}{x_h x_k}\sum_{h-q}^{k} x_i\cdots x_{i+q}+\frac{1}{x_h}\sum_{k+1}^{h} x_i\cdots x_{i+q}$$

$$-\frac{1}{x_k}\sum_{k-q}^{h-q-1} x_i\cdots x_{i+q}.$$

The proof of (ii) is symmetric to (i); hence it is not given here. ∎

<u>Lemma 1.2.2</u> Let $1\leq k\leq p-1$. Then

(i) If $x_k\geq 2$, $\sigma(\pi<k+1,k>)-\sigma(\pi)=$

$$\frac{x_k-x_{k+1}-1}{x_k x_{k+1}}\sum_{k+1-q}^{k} x_i\cdots x_{i+q}+x_{k+2}\cdots x_{k+1+q}-x_{k-q}\cdots x_{k-1}.$$

(ii) If $x_{k+1}\geq 2$, $\sigma(\pi<k,k+1)-\sigma(\pi)=$

$$\frac{x_{k+1}-x_k-1}{x_k x_{k+1}}\sum_{k+1-q}^{k} x_i\cdots x_{i+q}+x_{k-q}\cdots x_{k-1}-x_{k+2}\cdots x_{k+1+q}.$$

<u>Proof</u>. This lemma is a special case of lemma 1.2.1. ∎

<u>Lemma 1.2.3</u> Let the distinct integers h and k in the range $1\leq h,k\leq p$ be such that $|h-k|>q$. Let $\pi=(x_1,\ldots,x_p)\in\Pi$ be such that $x_k\geq 2$. Then $\sigma(\pi<h,k>)-\sigma(\pi)=$

$$\frac{1}{x_h}\sum_{h-q}^{h} x_i\cdots x_{i+q}-\frac{1}{x_k}\sum_{k-q}^{k} x_i\cdots x_{i+q}.$$

<u>Proof</u>. The proof involves breaking $\sigma(\pi<h,k>)$ into five

partial summations as in lemma 1.2.1. Because it is but a variation of the proof of lemma 1.2.1, it is omitted. ∎

<u>Lemma 1.2.4</u>

(i) $\sigma(\vec{\pi})-\sigma(\pi)=x_p x_1\ldots x_q - x_{p-q}\ldots x_p$.

(ii) $\sigma(\overleftarrow{\pi})-\sigma(\pi)= -x_1\ldots x_{1+q} + x_{p-q+1}\ldots x_p x_1$.

<u>Proof of (i)</u>.

$$\sigma(\vec{\pi}) = y_1\ldots y_{1+q}+\sum_{i=2}^{p-q} y_i\ldots y_{i+q}$$

$$= x_p x_1\ldots x_q+\sum_{i=1}^{p-q-1} x_i\ldots x_{i+q}$$

$$= x_p x_1\ldots x_q+\sum_{i=1}^{p-q} x_i\ldots x_{i+q}-x_{p-q}\ldots x_p$$

$$= \sigma(\pi)+x_p x_1\ldots x_q-x_{p-q}\ldots x_p.$$

The proof of (ii) is symmetric to (i); so it is omitted. ∎

The next lemma proves the existence of a certain partition needed in theorem 1.2.6.

<u>Lemma 1.2.5</u> Let $\pi\varepsilon\Pi$ be arbitrary but fixed. Let the fixed distinct integers h and k in the range $1\leq h<k\leq p$ be such that $q<k-h$. Then there exists a partition $\pi^\circ=(x_1^\circ,\ldots,x_p^\circ)\varepsilon\Pi$ such that

(i) $\sigma(\pi^\circ)\geq\sigma(\pi)$.

(ii) $x_k^\circ=1$ or $x_h^\circ=1$.

(iii) $x_i^\circ=x_i$ for $i\neq h,k$.

<u>Proof</u>. If in π, $x_h=1$ or $x_k=1$, let $\pi^\circ=\pi$ and the lemma is proved. So assume $x_h\geq 2$ and $x_k\geq 2$.

From lemma 1.2.3

$$\sigma(\pi<h,k>)-\sigma(\pi)=\frac{1}{x_h}\sum_{h-q}^{h} x_i\cdots x_{i+q}-\frac{1}{x_k}\sum_{k-q}^{k} x_i\cdots x_{i+q}$$

$$=-\left(-\frac{1}{x_h}\sum_{h-q}^{h} x_i\cdots x_{i+q}+\frac{1}{x_k}\sum_{k-q}^{k} x_i\cdots x_{i+q}\right)$$

$$=-\Big(\sigma(\pi<k,h>)-\sigma(\pi)\Big).$$

Thus $\sigma(\pi<h,k>)-\sigma(\pi)\geq 0$ or $\sigma(\pi<k,h>)-\sigma(\pi)\geq 0$.

<u>Case 1 $\sigma(\pi<h,k>)-\sigma(\pi)\geq 0$</u>. The existence of π° satisfying $x_k^\circ=1$ is proved by induction on x_k.

If $x_k=2$, let $\pi^\circ=\pi<h,k>$ and the conditions (i)-(iii) are satisfied. (In particular in (ii), $x_k^\circ=y_k=x_k-1=1$.)

Now assume $x_k=D$, and that π° satisfying (i), (iii) and $x_k^\circ=1$ exists for π' in which $x_k'=D-1$. We note that in $\pi'=\pi<h,k>$, $x_k'=D-1$. So the inductive hypothesis can be applied to $\pi<h,k>$ to produce π° for $\pi<h,k>$. We verify that π° satisfies (i)-(iii) for π as well.

(i) $\sigma(\pi^\circ)\geq\sigma(\pi<h,k>)$, by inductive hypothesis,

$\geq\sigma(\pi)$, by assumption for case 1.

(ii) $x_k^\circ=1$, by inductive hypothesis.

(iii) $x_i^\circ=y_i$ for $i\neq h,k$, by inductive hypothesis,

$= x_i$ for $i\neq h,k$, by definition of $\pi<h,k>$.

This completes the induction for this case.

<u>Case 2 $\sigma(\pi<k,h>)-\sigma(\pi)\geq 0$</u>. The proof is dual with case 1; so it is omitted. ▮

Theorem 1.2.6 There is a $\bar{\pi}^*$ in Π such that for some integers m and n in the range $1\leq m<n\leq p$ such that $m+q=n$, it is true that $\bar{x}_i^*=1$ for $i=1,\ldots,m-1,n+1,\ldots,p$.

Proof. Let $\bar{\pi}$ be fixed. Select r and s so that $\bar{x}_r\geq 2$ and $\bar{x}_i=1$ for $i=1,\ldots,r-1$; $\bar{x}_s\geq 2$ and $\bar{x}_i=1$ for $i=s+1,\ldots,p$.

The existence of $\bar{\pi}^*$ is proved by induction on the difference $d=s-r$ from 1 to $p-2$. (For $d=p-1$ the theorem is vacuous.)

For $d=1,\ldots,q$, let $\bar{\pi}^*=\bar{\pi}$, and the theorem is trivial for some suitable choice of m and n in the ranges $1\leq m\leq r$ and $s\leq n\leq p$.

Assume now that for any $\bar{\pi}^\circ$, in which r° and s° (defined similarly to r and s) are such that $s^\circ-r^\circ=d^\circ<d$, a partition $\bar{\pi}^*$ exists.

Let $q<d\leq p-2$ for $\bar{\pi}$. Applying lemma 1.2.5 to $\bar{\pi}$ with $h=r$ and $k=s$, we get a partition $\bar{\pi}^\circ$ satisfying (i)-(iii), lemma 1.2.5.

From (i) we have $\sigma(\bar{\pi}^\circ)\geq\sigma(\bar{\pi})=\mu$. And from (ii) and (iii) it follows that $d^\circ<d$. So we can apply the inductive hypothesis to $\bar{\pi}^\circ$ to produce $\bar{\pi}^*$. ∎

It is possible for π to be such that $\mu=\sigma(\pi)$, but that the condition in theorem 1.2.6 fails for π. For example, let $N=7$, $p=3$, $q=1$. It turns out that $\mu=12$. The partition (2,3,2) yields μ, but it does not satisfy theorem 1.2.6.

On the other hand, the partition (1,2,4) satisfies $x_i=1$ for $i=1,\ldots,m-1,n+1,\ldots,p$ for m and n such that $n-m=q=1$, but (1,2,4) fails to yield μ. The partitions (1,3,3) and (1,4,2) satisfy both conditions.

In theorem 1.2.7 and the two lemmas following it $\bar{\pi}$ designates a partition of the type whose existence is proved in theorem 1.2.6.

Theorem 1.2.7 There exists a partition $\bar{\pi}$ such that $|\bar{x}_i-\bar{x}_{i+1}|\leq 1$ for $i=m,\ldots,n-1$.

The theorem is implied by the next three lemmas.

Lemma 1.2.8 If in $\bar{\pi}$, $2\leq m$ and $n\leq p-1$, then $|\bar{x}_i-\bar{x}_{i+1}|\leq 1$ for $i=m,\ldots,n-1$.

Proof. Let π be such that $x_i=1$ for $i=1,\ldots,m-1,n+1,\ldots,p$ for some m and n such that $n-m=q$. But assume $|x_k-x_{k+1}|\geq 2$ for some k, $m\leq k\leq n-1$. We show $\mu\neq\sigma(\pi)$.

Case 1 $x_k-x_{k+1}\geq 2$. By lemma 1.2.2, $\sigma(\pi<k+1,k>)-\sigma(\pi)=$

$$\frac{x_k-x_{k+1}-1}{x_kx_{k+1}}\sum_{k+1-q}^{k} x_i\cdots x_{i+q}+x_{k+2}\cdots x_{k+1-q}-x_{k-q}\cdots x_{k-1}$$

$$\geq \frac{1}{x_kx_{k+1}}\sum_{k+1-q}^{k} x_i\cdots x_{i+q}+x_{k+2}\cdots x_{k+1-q}-x_{k-q}\cdots x_{k-1}.$$

If $k-q<1$, then $x_{k-q}\cdots x_{k-1}=0$. Hence

$$\sigma(\pi<k+1,k>)-\sigma(\pi)\geq \frac{x_1\cdots x_{1+q}}{x_kx_{k+1}}>0.$$

If $k-q\geq 1$, then $\sigma(\pi<k+1,k>)-\sigma(\pi)\geq$

$$\frac{x_{k+1-q}\cdots x_{k+1}}{x_k x_{k+1}}+\frac{x_{k+2-q}\cdots x_{k+2}}{x_k x_{k+1}}-x_{k-q}\cdots x_{k-1}$$

$$=\frac{1}{x_k x_{k+1}}(x_{k+1-q}\cdots x_{k+1}+x_{k+2-q}\cdots x_{k+2}-x_{k-q}\cdots x_{k+1})$$

$$=\frac{x_{k+2-q}\cdots x_{k+2}}{x_k x_{k+1}},\text{ because } x_{k-q}=1,$$

>0, because $k+2\leq p$.

In either case $\mu\geq\sigma(\pi<k+1,k>)>\sigma(\pi)$.

Hence $\mu\neq\sigma(\pi)$. This implies the lemma.

Case 2 $x_{k+1}-x_k\geq 2$. The argument is dual to case 1. Hence it is omitted. ∎

Lemma 1.2.9 If in $\bar{\pi}$ either $m=1$ or $n=p$ (but not both), then there exists a partition $\bar{\pi}'$ such that $|\bar{x}_i'-\bar{x}_{i+1}'|\leq 1$ for $i=m,\ldots,n-1$.

Proof. The proof is broken into two symmetric cases.

Case 1 $m\geq 2$ and $n=p$. If $m\leq p-2$, then using the same argument as in lemma 1.2.8 it can be established that $|\bar{x}_i-\bar{x}_{i+1}|\leq 1$ for $i=m,\ldots,p-2$.

Now assume that $\bar{x}_{p-1}<\bar{x}_p$. We show $\mu\neq\sigma(\bar{\pi})$, a contradiction. From lemma 1.2.2, $\sigma(\bar{\pi}<p-1,p>)-\sigma(\bar{\pi})=$

$$\frac{\bar{x}_p-\bar{x}_{p-1}-1}{\bar{x}_{p-1}\bar{x}_p}\sum_{i=p-q}^{p-1}\bar{x}_i\cdots\bar{x}_{i+q}+\bar{x}_{p-1-q}\cdots\bar{x}_{p-2}-\bar{x}_{p+1}\cdots\bar{x}_{p+q}$$

$\geq\bar{x}_{p-1-q}\cdots\bar{x}_{p-2}$

>0.

Hence $\mu\neq\sigma(\pi)$. This implies that in $\bar{\pi}$, $\bar{x}_{p-1}\geq\bar{x}_p$.

Now we prove $\bar{x}_{p-1} \leq \bar{x}_p + 2$ in $\bar{\pi}$. For assume $\bar{x}_{p-1} \geq \bar{x}_p + 3$. We show $\mu \neq \sigma(\bar{\pi})$. By lemma 1.2.2, $\sigma(\bar{\pi}\langle p,p-1\rangle) - \sigma(\bar{\pi}) =$

$$\frac{\bar{x}_{p-1}-\bar{x}_p-1}{\bar{x}_{p-1}\bar{x}_p} \sum_{i=p-q}^{p-1} \bar{x}_i \cdots \bar{x}_{i+q} + \bar{x}_{p+1} \cdots \bar{x}_{p+q} - \bar{x}_{p-1-q} \cdots \bar{x}_{p-2}$$

$$\geq \frac{2}{\bar{x}_{p-1}\bar{x}_p} \bar{x}_{p-q} \cdots \bar{x}_p - \bar{x}_{p-1-q} \cdots \bar{x}_{p-2}.$$

But $\bar{x}_{p-1-q} = 1$. So $\sigma(\bar{\pi}\langle p,p-1\rangle) - \sigma(\bar{\pi}) \geq \frac{\bar{x}_{p-q} \cdots \bar{x}_p}{\bar{x}_{p-1}\bar{x}_p} > 0$.

So $\mu \neq \sigma(\bar{\pi})$. This contradiction establishes that $\bar{x}_{p-1} \leq \bar{x}_p + 2$ in $\bar{\pi}$.

We have shown so far that in $\bar{\pi}$ it is true that $|\bar{x}_i - \bar{x}_{i+1}| \leq 1$ for $i=m,\ldots,p-2$, and $\bar{x}_{p-1} - \bar{x}_p = 0, 1$, or 2. If $\bar{x}_{p-1} - \bar{x}_p = 0$ or 1 the lemma is proved.

If $\bar{x}_{p-1} - \bar{x}_p = 2$, let $\bar{\pi}' = \bar{\pi}\langle p,p-1\rangle$. Then by lemma 1.2.2,

$$\sigma(\bar{\pi}') - \sigma(\bar{\pi}) = \frac{\bar{x}_{p-1}-\bar{x}_p-1}{\bar{x}_{p-1}\bar{x}_p} \sum_{i=p-q}^{p-1} \bar{x}_i \cdots \bar{x}_{i+q} + \bar{x}_{p+1} \cdots \bar{x}_{p+q} - \bar{x}_{p-1-q} \cdots \bar{x}_{p-2}.$$

$$\geq \frac{\bar{x}_{p-q} \cdots \bar{x}_p}{\bar{x}_{p-1}\bar{x}_p} - \bar{x}_{p-1-q} \cdots \bar{x}_{p-2}$$

$$= 0, \text{ because } \bar{x}_{p-1-q} = 1.$$

So $\mu = \sigma(\bar{\pi}) \leq \sigma(\bar{\pi}')$. Hence $\mu = \sigma(\bar{\pi}')$. Consequently it must be that $|\bar{x}'_i - \bar{x}'_{i+1}| \leq 1$ for $i=m,\ldots,p-2$. And $|\bar{x}'_{p-1} - \bar{x}'_p| = 0 \leq 1$. This proves the lemma for this case.

<u>Case 2 $m=1$ and $n \leq p-1$</u>. The proof is symmetric to case 1; so it is not given. ▮

In the example given earlier, the partition $\bar{\pi} = (1,4,2)$ is an example of a partition for which it is actually necessary to carry out the construction in the proof above to obtain the partition $\bar{\pi}' = (1,3,3)$.

Lemma 1.2.10 If $\mu=\sigma(\pi)$ and $p\leq 1+2q$, then $|x_i-x_j|\leq 1$ for all i,j such that $p-q\leq i,j\leq 1+q$.

Proof. Let π be such that $x_k-x_h\geq 2$ for some h and k in the range $p-q\leq h,k\leq 1+q$. We show $\mu\neq\sigma(\pi)$.

Case 1 $k<h$. By lemma 1.2.1, $\sigma(\pi<h,k>)-\sigma(\pi)=$

$$\frac{x_k-x_h-1}{x_hx_k}\sum_{h-q}^{k} x_i\cdots x_{i+q}+\frac{1}{x_h}\sum_{k+1}^{h} x_i\cdots x_{i+q}-\frac{1}{x_k}\sum_{k-q}^{h-q-1} x_i\cdots x_{i+q}$$

$$\geq\frac{1}{x_hx_k}\sum_{h-q}^{k} x_i\cdots x_{i+q}$$

$$\geq\frac{x_1\cdots x_{1+q}}{x_hx_k}$$

$$>0.$$

So $\mu\neq\sigma(\pi)$. This proves the lemma for this case.

Case 2 $h<k$. The proof is not given, for it is dual to case 1.∎

A special case of lemma 1.2.10 is the case $m=1$ and $n=p$ in the partition $\bar{\pi}$ of theorem 1.2.7. In this case $|\bar{x}_i-\bar{x}_j|\leq 1$ for $1=p-q\leq i,j\leq 1+q=p$. This special case and lemmas 1.2.8 and 1.2.9 give us immediately theorem 1.2.7.∎

The next four lemmas are preparatory for the next main result, theorem 1.2.15.

Lemma 1.2.11 Let $p\leq 1+2q$. Let $\pi\varepsilon\Pi$ and $\pi'\varepsilon\Pi$ be such that $(x'_{p-q},\ldots,x'_{1+q})$ is a permutation of $(x_{p-q},\ldots,x_{1+q})$, and $x'_i=x_i$ for $i=1,\ldots,p-q-1,2+q,\ldots,p$. Then $\sigma(\pi')=\sigma(\pi)$.

Proof.

$$\sigma(\pi) = \sum_{i=1}^{p-q} x_i \cdots x_{i+q}$$

$$= x_{p-q} \cdots x_{1+q} \sum_{i=1}^{p-q} x_i \cdots x_{p-q-1} x_{2+q} \cdots x_{i+q}$$

$$= x'_{p-q} \cdots x'_{1+q} \sum_{i=1}^{p-q} x'_i \cdots x'_{p-q-1} x'_{2+q} \cdots x'_{i+q}$$

$$= \sum_{i=1}^{p-q} x'_i \cdots x'_{i+q}$$

$$= \sigma(\pi'). \blacksquare$$

Lemma 1.2.12 If $\mu=\sigma(\pi)$ and $1+2q\leq p$, then $x_1 \leq \cdots \leq x_{1+q}$ and $x_{p-q} \geq \cdots \geq x_p$.

Proof. Assume that $x_k > x_{k+1}$ for some k in the range $1\leq k\leq q$. We show $\mu\neq\sigma(\pi)$. By lemma 1.2.2, $\sigma(\pi<k+1,k>)-\sigma(\pi)=$

$$\frac{x_k - x_{k+1} - 1}{x_k x_{k+1}} \sum_{i=k+1-q}^{k} x_i \cdots x_{i+q} + x_{k+2} \cdots x_{k+1+q} - x_{k-q} \cdots x_{k-1}$$

$$\geq x_{k+2} \cdots x_{k+1+q}$$

$$> 0.$$

So $\mu\neq\sigma(\pi)$. This establishes that if $\mu=\sigma(\pi)$, then $x_1 \leq \cdots \leq x_{1+q}$.

A similar argument proves that $x_{p-q} \geq \cdots \geq x_p$. $\blacksquare$

Lemma 1.2.13 If $\mu=\sigma(\pi)$ and $p\leq 1+2q$, then $x_1 \leq \cdots \leq x_{p-q}$ and $x_{1+q} \geq \cdots \geq x_p$.

Proof. The proof is in the same spirit as the proof of lemma 2.2.12. So we omit it. $\blacksquare$

<u>Lemma 1.2.14</u> Let $p\le 1+2q$. If $\mu=\sigma(\pi)$ then

(i) $x_i\le x_j$ for $i=1,\ldots,p-q-1$ and $j=p-q,\ldots,1+q$,

(ii) $x_j\ge x_i$ for $i=2+q,\ldots,p$ and $j=p-q,\ldots,1+q$.

<u>Proof of (i)</u>. Let π be such that $x_k>x_h$ for some h and k in the ranges $1\le k\le p-q-1$ and $p-q\le h\le 1+q$. We show $\mu\ne\sigma(\pi)$.

By lemma 1.2.1, $\sigma(\pi\langle h,k\rangle)-\sigma(\pi)=$

$$\frac{x_k-x_h-1}{x_h x_k}\sum_{h-q}^{k} x_i\cdots x_{i+q}+\frac{1}{x_h}\sum_{k+1}^{h} x_i\cdots x_{i+q}-\frac{1}{x_k}\sum_{k-q}^{h-q-1} x_i\cdots x_{i+q}$$

$$\ge\frac{1}{x_h}\sum_{k+1}^{h} x_i\cdots x_{i+q}$$

$$>\frac{x_{p-q}\cdots x_p}{x_h}$$

$$>0.$$

So $\mu\ne\sigma(\pi)$.

A similar argument establishes (ii).∎

For reference in the next two theorems we state five conditions for partitions in Π.

(1) $\mu=\sigma(\pi)$.

(2) $x_i=1$ for $i=1,\ldots,m-1,n+1,\ldots,p$, where $n-m=q$.

(3) $|x_i-x_{i+1}|\le 1$ for $i=m,\ldots,n-1$.

(4) $x_m\le\cdots\le x_c\ge\cdots\ge x_n$ for some c, $m\le c\le n$.

(5) $|x_i-x_j|\le 2$ for $i=m,\ldots,n$ and $j=m,\ldots,n$.

Theorems 1.2.6 and 1.2.7 guarantee the existence of a π satisfying (1)-(3). We show next there is a π satisfying (1)-(4).

<u>Theorem 1.2.15</u> There is a partition $\pi\in\Pi$ satisfying (1)-(4).

Proof. We consider two main cases, $p\leq 1+2q$ and $p\geq 1+2q$. The case $p\leq 1+2q$ is further divided into subcases.

Case 1 $p\leq 1+2q$. Throughout this case π is a partition satisfying (1) and (2).

Case 1a $m=1$, $n=p$. Define π' by permuting $x_1,\ldots,x_p$ so that $x'_1\leq\ldots\leq x'_c\geq\ldots\geq x'_p$ for some c, $1\leq c\leq p$. By lemma 1.2.11 $\sigma(\pi')=\sigma(\pi)=\mu$. Now π' satisfies (2) vacuously. By lemma 1.2.10, π' satisfies (3).

Case 1b $m=1$ or $n=p$, but not both. We consider first the case $m=1$ and $n\leq p-1$.

Let π satisfy (1)-(3). Now define π' by permuting $x_{p-q},\ldots,x_{1+q}$ so that $x'_{p-q}\leq\ldots\leq x'_{1+q}$ and by letting $x'_i=x_i$ for $i=1,\ldots,p-q-1,2+q,\ldots,p$. We show π' is the desired partition.

By lemma 1.2.11, $\sigma(\pi')=\sigma(\pi)=\mu$. And by definition $x'_i=x_i=1$ for $i=2+q,\ldots,p$. So π' satisfies (1) and (2).

Now $x'_1\leq\ldots\leq x'_{p-q}$ by lemma 1.2.13 and $x'_{p-q}\leq\ldots\leq x'_{1+q}$ by definition of π'. So for $c=1+q$, $x'_1=x'_m\leq\ldots\leq x'_c=x'_{1+q}=x'_n$, i.e. (4) is true in π'.

Finally, for $i=1,\ldots,p-q-2$, $|x'_i-x'_{i+1}|=|x_i-x_{i+1}|\leq 1$. And for $i=p-q,\ldots,q$, $|x'_i-x'_{i+1}|\leq 1$ by lemma 1.2.10. Now

$$
\begin{aligned}
x_{p-q-1} &\leq x'_{p-q-1} \\
&\leq x'_{p-q}, \text{ by lemma 1.2.14}, \\
&\leq x_{p-q}.
\end{aligned}
$$

So from $|x_{p-q-1}-x_{p-q}|\leq 1$ it follows that $|x'_{p-q-1}-x'_{p-q}|\leq 1$. So π' satisfies (3).

If π satisfies (1)-(3) and $m \geq 2$, $n=p$, then $\hat{\pi}$ satisfies (1)-(3) also, and the above proof can be applied to $\hat{\pi}$.

Case 1c $m \geq 2$, $n \leq p-1$. Define π' by permuting $x_{p-q}, \ldots, x_{1+q}$ so that $x'_{p-q} \leq \ldots \leq x'_c \geq \ldots \geq x'_{1+q}$ and such that $x'_i = x_i$ for $i=1,\ldots,p-q-1,2+q,\ldots,p$. By lemma 1.2.11, $\sigma(\pi')=\sigma(\pi)=\mu$. By lemma 1.2.13, $x'_1 \leq \ldots \leq x'_{p-q}$ and $x'_{1+q} \geq \ldots \geq x'_p$. So $x'_m \leq \ldots \leq x'_c \geq \ldots \geq x'_n$ in π'. Because $x'_i = x_i$ for $i=1,\ldots,p-q-1,2+q,\ldots,p$ (and hence for $i=1,\ldots,m-1,n+1,\ldots,p$), we have $x'_i = x_i = 1$ for $i=1,\ldots,m-1,n+1,\ldots,p$. Consequently, π' satisfies (2). Finally, π' satisfies (3) by lemma 1.2.8.

We note that in cases 1a-1c, it is possible to choose c so that $1+q \leq c$.

Case 2 $p \geq 1+2q$. Let π satisfy (1)-(3). To avoid the trivial cases, assume $m>1$, $n<p$, and assume that there are two non-adjacent non-ones in $x_m, \ldots, x_n$.

For $i=m,\ldots,n-1$, define ϕ_i by

$$\phi_i = x_{i-q} \cdots x_{i-1} - x_{i+2} \cdots x_{i+1+q}.$$

Now
$$\begin{aligned}\phi_m &= x_{m-q} \cdots x_{m-1} - x_{m+2} \cdots x_{m+1+q} \\ &\leq 1-2 \\ &< 0.\end{aligned}$$

And
$$\begin{aligned}\phi_{n-1} &= x_{n-1-q} \cdots x_{n-2} - x_{n+1} \cdots x_{n+q} \\ &\geq 2-1 \\ &> 0.\end{aligned}$$

For $m \leq i \leq n-2$,

$$\phi_{i+1} - \phi_i = x_{i+1-q} \cdots x_i - x_{i+3} \cdots x_{i+2+q} - x_{i-q} \cdots x_{i-1} + x_{i+2} \cdots x_{i+1+q}$$

$$= (x_i - x_{i-q}) \frac{x_{i-q} \cdots x_i}{x_{i-q} x_i} + (x_{i+2} - x_{i+2+q}) \frac{x_{i+2} \cdots x_{i+2+q}}{x_{i+2} x_{i+2+q}} .$$

≥ 0.

Hence $\phi_m \leq \cdots \leq \phi_{n-1}$.

Let r be such that $\phi_r < 0$ and $\phi_{r+1} \geq 0$. Let s be such that $\phi_{s-1} \leq 0$ and $\phi_s > 0$. Then $m \leq r < s \leq n$.

We now show that $x_m \leq \cdots \leq x_{r+1}$. For assume that for some fixed k, $m \leq k \leq r$, it is true that $x_k > x_{k+1}$. Then $x_k = x_{k+1} + 1$ because π satisfies (3).

Then by lemma 1.2.2, $\sigma(\pi\langle k+1,k\rangle) - \sigma(\pi) =$

$$\frac{(x_k - x_{k+1} - 1)}{x_k x_{k+1}} \sum_{k+1-q}^{k} x_i \cdots x_{i+q} - x_{k-q} \cdots x_{k-1} + x_{k+2} \cdots x_{k+1+q}$$

$= -x_{k-q} \cdots x_{k-1} + x_{k+2} \cdots x_{k+1+q}$

$= -\phi_k$

$\geq -\phi_r$

> 0

So $\mu \neq \sigma(\pi)$. But this contradicts the choice of π. So $x_m \leq \cdots \leq x_{r+1}$.

A similar argument proves that $x_s \geq \cdots \geq x_n$.

Next we show that $\phi_i = 0$ for at most one ϕ_i in the sequence $\phi_m, \ldots, \phi_{n-1}$. Assume now that $\phi_j = 0$. Necessarily $\phi_{j-q} \cdots x_{j-1}$ and $x_{j+2} \cdots x_{j+1+q}$ both vanish or both are non-vanishing and equal. If $x_{j-q} \cdots x_{j-1}$ and $x_{j+2} \cdots x_{j+1+q}$ both vanish, it follows that $p \leq 2q$, which is not under consideration. Therefore it must be that $x_{j-q} \cdots x_{j-1}$ and $x_{j+2} \cdots x_{j+1+q}$ are equal and nonvanishing.

Assume further that $\phi_{j+1}=0$. Then

$$
\begin{aligned}
0&=\phi_{j+1}\\
&=\frac{x_j}{x_{j-q}}\,x_{j-q}\cdots x_{j-1}-\frac{x_{j+2+q}}{x_{j+2}}\,x_{j+2}\cdots x_{j+1+q}\\
&=\frac{x_j}{x_{j-q}}-\frac{x_{j+2+q}}{x_{j+2}}\\
&=x_j-\frac{1}{x_{j+2}}.
\end{aligned}
$$

Consequently $x_j=x_{j+2}=1$.

If only ϕ_j and ϕ_{j+1} vanish, then $x_m\leq\cdots\leq x_j = 1=x_{j+2}\geq\cdots\geq x_n$. So only x_{j+1} is a non-one, contradicting the choice of π as containing two non-ones. If there are three or more elements in $\phi_m,\ldots,\phi_{n-1}$ which vanish, a similar argument proves that $x_m=\ldots=x_n=1$, again a contradiction.

Thus we have $s\leq r+2$, and $x_m\leq\cdots\leq x_{r+1}, x_{r+2}\geq\cdots\geq x_n$. Let $x_c=\max\,(x_{r+1},x_{r+2})$. Then $x_m\leq\cdots\leq x_c\geq\cdots\geq x_n$. ∎

Theorem 1.2.16 If π satisfies (1)-(4), then π satisfies (5) also.

Proof. Let π satisfy (2)-(4) but not (5). We show $\mu\neq\sigma(\pi)$.

Assume without loss of generality that $x_m\leq x_n$, for otherwise the following proof can be applied to $\hat{\pi}$. Now define h and k by

$x_i=x_m$ for $i=m,\ldots,h$ and $x_{h+1}\neq x_m$;

$x_k\geq x_h+3$ and $x_i<x_h+3$ for $i=k+1,\ldots,n$.

We note that $h<c\leq k$ and that $x_i>x_h$ for $i=h+1,\ldots,k-1$.

By lemma 1.2.1, $\sigma(\pi\langle h,k\rangle)-\sigma(\pi)=$

$$\frac{x_k-x_h-1}{x_h x_k}\sum_{k-q}^{h} x_i\cdots x_{i+q}+\frac{1}{x_h}\sum_{h-q}^{k-q-1} x_i\cdots x_{i+q}-\frac{1}{x_k}\sum_{h+1}^{k} x_i\cdots x_{i+q}$$

$$>\frac{2}{x_h x_k}\,x_m\cdots x_n-\frac{1}{x_k}\sum_{h+1}^{k} x_i\cdots x_{i+q}$$

$$\geq \frac{2}{x_k}\,x_{h+1}\cdots x_n-\frac{1}{x_k}\sum_{h+1}^{k} x_i\cdots x_{i+q}.$$

Now $\dfrac{x_{h+1}\cdots x_n}{x_k}-\dfrac{x_h}{x_k}\sum_{h+2}^{k} x_i\cdots x_{i+q}-\dfrac{x_k\cdots x_n}{x_k}=$

$$\frac{x_{h+1}\cdots x_n}{x_k}-\frac{x_k\cdots x_n}{x_k}\sum_{i=h+2}^{k} x_h\frac{x_i\cdots x_k}{x_k}-\frac{x_k\cdots x_n}{x_k}$$

$$=\frac{x_k\cdots x_n}{x_k}\{x_{h+1}\cdots x_{k-1}-x_{h+2}\cdots x_{k-1}x_h-x_{h+3}\cdots x_{k-1}x_h-\cdots-x_{k-1}x_h-x_h-1\}$$

$$=\frac{x_k\cdots x_n}{x_k}\{(x_{k-1}(x_{k-2}(x_{k-3}\cdots(x_{h+2}(x_{h+1}-x_h)-x_h)\cdots-x_h)-x_h)-x_h)-1\}$$

≥ 0, because $x_i>x_h$ for $i=h+1,\ldots,k-1$. Thus

$$\frac{x_{h+1}\cdots x_n}{x_k}\geq\frac{x_h}{x_k}\sum_{h+2}^{k} x_i\cdots x_{i+q}+\frac{x_k\cdots x_n}{x_k}$$

$$>\frac{x_h}{x_k}\sum_{h+2}^{k} x_i\cdots x_{i+q}.$$

So now $\dfrac{1}{x_k}\sum_{h+1}^{k} x_i\cdots x_{i+q}=\dfrac{x_{h+1}\cdots x_n}{x_k}+\dfrac{1}{x_k}\sum_{h+2}^{k} x_i\cdots x_{i+q}$

$$<\frac{x_{h+1}\cdots x_n}{x_k}+\frac{1}{x_h}\,\frac{x_{h+1}\cdots x_n}{x_k}$$

$$=\frac{x_{h+1}\cdots x_n}{x_k}\cdot\frac{x_h+1}{x_h}$$

Using these results in $\sigma(\pi')-\sigma(\pi)$ gives finally

$$\sigma(\pi')-\sigma(\pi) \geq 2\frac{x_{h+1}\cdots x_n}{x_k} - \frac{1}{x_k}\sum_{h+1}^{k} x_i\cdots x_{i+q}$$

$$> 2\frac{x_{h+1}\cdots x_n}{x_k} - \frac{x_h+1}{x_h}\cdot\frac{x_{h+1}\cdots x_n}{x_k}$$

$$\geq 0.$$

So $\mu\neq\sigma(\pi)$. ▮

For reference through theorem 1.2.19, we state some conditions for partitions in π.

(1) $\mu=\sigma(\pi)$.

(2) $x_i=1$ for $i=1,\ldots,m-1,n+1,\ldots,p$, where $n-m=q$.

(3) $|x_i-x_{i+1}|\leq 1$ for $i=m,\ldots,n-1$ and

$|x_i-x_j|\leq 2$ for $m\leq i,j\leq n$.

(4) $x_m\leq\ldots\leq x_c\geq\ldots\geq x_n$ for some c, $1+q\leq c\leq n$.

(5) $2\leq x_m\leq x_n$.

(6) $x_1\ldots x_q<x_{p-q}\ldots x_p$.

A partition satisfying (1)-(6) is denoted by $\bar{\pi}$.

Furthermore, if π satisfies (2)-(6), the numbers r,s, and t are defined as follows: If $x_i\neq x_j$ for some i and j, $m\leq i,j\leq n$, then r is such that

$x_i=x_m$ for $i=m,\ldots,r$ and $x_{r+1}=x_m+1$.

If $|x_i-x_j|>1$ for some i and j, $m\leq i,j\leq n$, s and t are such that

$x_i=x_m+1$ for $i=r+1,\ldots,s$ and $x_{s+1}=x_m+2$.

$x_i=x_m+2$ for $i=s+1,\ldots,t$ and $x_{t+1}\neq x_m+2$.

The requirement $x_m\leq x_n$ is actually no real restriction, for otherwise $\hat{\pi}$ can be considered. However, $2\leq x_m$ is a definite restriction. We treat separately the special case in which some of the numbers $x_m,\ldots,x_n$ are one, and for the next several theorems we assume $x_i \geq 2$ for $i=m,\ldots,n$.

In (4) the selection of c in the range $1+q \leq c \leq n$ is possible, as noted at the end of case 1, theorem 1.2.15. The significance of this choice for c is that if $|x_i-x_j|>1$ for some i and j, $m\leq i,j\leq n$, then $1+q\leq c\leq t$.

To assert the existence of a partition $\vec{\pi}$, we now need only prove we can construct a partition satisfying (1)-(6) from one satisfying (1)-(5). This we do in the next lemma.

Lemma 1.2.17 If π satisfies (1)-(5), then there exists a π' satisfying (1)-(6).

Proof. If $p=q+1$, the lemma is trivially true. So assume that $p>q+1$. Assume now that in π, $x_1\cdots x_q \geq x_{p-q}\cdots x_p$. Then it must be true that $n<p$ and $x_p=1$.

By lemma 1.2.4,

$$\begin{aligned}\sigma(\vec{\pi})-\sigma(\pi) &= x_p x_1\cdots x_q - x_{p-q}\cdots x_p \\ &= x_1\cdots x_q - x_{p-q}\cdots x_p \\ &\geq 0.\end{aligned}$$

So $\mu=\sigma(\vec{\pi})$.

The sequence $x_m,\ldots,x_n$ is untouched in $\vec{\pi}$ except for a shift to the right; so it is immediate that (2)-(5) hold in $\vec{\pi}$.

A sufficient number of repetitions of this shifting to the right will produce a π' satisfying (1)-(5), and such that $x'_1 \cdots x'_q < x'_{p-q} \cdots x'_p$, if for no other reason than that $(x_m, \ldots, x_n)$ will eventually be shifted to the position $x'_{p-q} \cdots x'_p$, in which case $x'_p = x_n \geq 2$ and $1 < p-q$ guarantee that $x'_1 \cdots x'_q < x'_{p-q} \cdots x'_p$. ∎

The next goal is to produce a partition $\bar{\pi}$ such that $|\bar{x}_i - \bar{x}_j| \leq 1$ for all i and j in the range $m \leq i,j \leq n$. Unfortunately, there are partitions satisfying (1)-(6) in which this condition fails. We give an example of one after the next lemma, which states several sufficient conditions for $|\bar{x}_i - \bar{x}_j| \leq 1$.

<u>Lemma 1.2.18</u> Assume that in $\bar{\pi}$, $|\bar{x}_i - \bar{x}_j| = 2$ for some i and j in the range $m \leq i,j \leq n$. Then

(i) $\bar{x}_m < \bar{x}_n$.

(ii) $r=m$.

(iii) $1+q=t$.

<u>Proof</u>. From lemma 1.2.10 the lemma is vacuously true if $p=q+1$. So assume without loss of generality that $p>1+q$.

Throughout this proof π is a partition satisfying (2)-(6) and $|x_i - x_j| = 2$ for some i and j in the range $m \leq i,j \leq n$.

By lemma 1.2.1, $\sigma(\pi\langle r,t\rangle) - \sigma(\pi) =$

$$\frac{x_t - x_r - 1}{x_r x_t} \sum_{t-q}^{r} x_i \cdots x_{i+q} + \frac{1}{x_r} \sum_{r-q}^{t-q-1} x_i \cdots x_{i+q} - \frac{1}{x_t} \sum_{r+1}^{t} x_i \cdots x_{i+q}$$

$$> \frac{1}{x_r x_t} \sum_{t-q}^{r} x_i \cdots x_{i+q} + \frac{1}{x_r} \sum_{r-q}^{t-q-1} x_i \cdots x_{i+q} - \frac{x_r + 1}{x_r} \cdot \frac{x_{r+1} \cdots x_n}{x_t},$$

as in lemma 1.2.16.

<u>Proof of (i)</u>. Assume that $x_m=x_n$. We show $\mu\neq\sigma(\pi)$. For

$\sigma(\pi<r,t>)-\sigma(\pi) >$

$$\frac{x_{r+1}\cdots x_n}{x_t} + \frac{x_{r+1}\cdots x_{n-1}}{x_t} - \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$$=\frac{x_{r+1}\cdots x_n}{x_t}\cdot\frac{x_r+1}{x_r} - \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}, \text{ because } x_m=x_n=x_r.$$

$=0.$

So $\mu\neq\sigma(\pi)$.

<u>Proof of (ii)</u>. Assume $r>m$. We show $\mu\neq\sigma(\pi)$.

$$\sigma(\pi<r,t>)-\sigma(\pi)>\frac{x_{m+1}\cdots x_n}{x_r x_t} + \frac{x_m\cdots x_n}{x_r x_t} - \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$$\geq \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t} - \frac{x_{r+1}}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$= 0.$

So $\mu\neq\sigma(\pi)$.

<u>Proof of (iii)</u>. From condition (4), $1+q\leq t$. Assume that $1+q<t$. From (i), $x_i=x_r+1$ for $i=t+1,\ldots,n$. We show $\mu\neq\sigma(\pi)$.

$\sigma(\pi<r,t>)-\sigma(\pi) >$

$$\frac{1}{x_r x_t}\sum_{t-q}^{r} x_i\cdots x_{i+q} + \frac{1}{x_r}\sum_{r-q}^{t-q-1} x_i\cdots x_{i+q} - \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$$\geq x_{r+1}\cdots x_{t-1}\sum_{i=t}^{n}\frac{x_t\cdots x_i}{x_t} + x_{r+1}\cdots x_{t-1} - \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$$=x_{r+1}\cdots x_{t-1}\left(\sum_{i=0}^{n-t}\frac{x_t\cdots x_n}{x_t(x_r+1)^i} + 1\right) - \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$$=\frac{x_{r+1}\cdots x_n}{x_t}\left(\sum_{i=0}^{n-t}\frac{1}{(x_r+1)^i} + \frac{1}{(x_r+1)^{n-t}}\right) - \frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$$=\frac{x_{r+1}\cdots x_n}{x_t}\left(\frac{(x_r+1)^{n-t+1}-1}{(x_r+1)^{n-t+1}}\cdot\frac{x_r+1}{x_r}+\frac{x_r}{(x_r+1)^{n-t+1}}\quad\frac{x_r+1}{x_r}\right)$$

$$-\frac{x_r+1}{x_r}\cdot\frac{x_{r+1}\cdots x_n}{x_t}$$

$$=\frac{x_{r+1}\cdots x_n}{x_t}\cdot\frac{x_r+1}{x_r}\left(\frac{(x_r+1)^{n-t+1}-1+x_r-(x_r+1)^{n-t+1}}{(x_r+1)^{n-t+1}}\right)$$

≥ 0.

So $\mu\neq\sigma(\pi)$. ∎

We present the following example to show that lemma 1.2.18 is not vacuous. Let $N=19$, $p=8$, and $q=4$. It turns out that $\mu=576$. Let $\bar{\pi}$ be given by $\bar{\pi}=(1,2,3,4,4,3,1,1)$. Then $\bar{\pi}$ satisfies (1)-(6). But $|\bar{x}_i-\bar{x}_j|=2$ for $i=2$, $j=5$. Necessarily $\bar{\pi}$ also satisfies (i)-(iii) of lemma 1.2.18. The partition $\bar{\pi}'$ produced in the next theorem is $\bar{\pi}'=(1,3,3,4,3,3,1,1)$.

<u>Theorem 1.2.19</u> There exists a partition $\bar{\pi}'$ such that $|\bar{x}_i'-\bar{x}_j'|\leq 1$ for all i and j in the range $m\leq i,j\leq n$.

<u>Proof</u>. Let $\bar{\pi}$ be such that $|\bar{x}_i-\bar{x}_j|=2$ for some i and j, $m\leq i,j\leq n$. Let $\bar{\pi}'=\bar{\pi}\langle m,t\rangle$. Evidently, $\bar{\pi}'$ satisfies (2)-(6).

From lemma 1.2.18, $|\bar{x}_i-\bar{x}_j|=2$ only when $i=m$. So in $\bar{\pi}'$, $|\bar{x}_i'-\bar{x}_j'|\leq 1$ for all i and j, $m\leq i,j\leq n$. To complete the proof we show $\mu=\sigma(\bar{\pi}')$.

We now prove $m\geq p-n$ in three cases.

<u>Case 1 $p-q\leq s$</u>.

$$\bar{x}_m(\bar{x}_m+1)^{s-m}(\bar{x}_m+2)^{t-s-1}=\bar{x}_1\cdots\bar{x}_q$$

$$< \bar{x}_{p-q} \cdots \bar{x}_p$$
$$= (\bar{x}_m+1)^{s-p+q+1}(\bar{x}_m+2)^{t-s-1}(\bar{x}_m+1)^{n-t}.$$

Hence $\bar{x}_m(\bar{x}_m+1)^{s-m} < (\bar{x}_m+1)^{s-p+q+1+n-t}$.

So $\left(\log_{(\bar{x}_m+1)} \bar{x}_m\right)+s-m < s-p+q+1+n-t$.

Consequently $m > p-q-1-n+t$.

Because $t-q-1=0$, $m > p-n$.

Case 2 $s<p-q\leq t$.

$$\bar{x}_m(\bar{x}_m+1)^{p-q-m-1}(\bar{x}_m+2)^{t-p+q}=$$
$$\bar{x}_m(\bar{x}_m+1)^{s-m}(\bar{x}_m+1)^{-s-1+p-q}(\bar{x}_m+2)^{t-p+q}$$
$$<\bar{x}_m(\bar{x}_m+1)^{s-m}(\bar{x}_m+2)^{-s-1+p-q}(\bar{x}_m+2)^{t-p+q}$$
$$=\bar{x}_m(\bar{x}_m+1)^{s-m}(\bar{x}_m+2)^{t-s-1}$$
$$=\bar{x}_1 \cdots \bar{x}_q$$
$$<\bar{x}_{p-q} \cdots \bar{x}_p$$
$$=(\bar{x}_m+2)^{t-p+q}(\bar{x}_m+1)^{n-t}$$

Hence $p-q-m-1<n-t$. It follows that $m>p-n$.

Case 3 $t<p-q$.

$$\bar{x}_m(\bar{x}_m+1)^{t-m-1}=\bar{x}_m(\bar{x}_m+1)^{s-m}(\bar{x}_m+1)^{t-s-1}$$
$$\leq \bar{x}_m(\bar{x}_m+1)^{s-m}(\bar{x}_m+2)^{t-s-1}$$
$$=\bar{x}_1 \cdots \bar{x}_q$$
$$<\bar{x}_{p-q} \cdots \bar{x}_p$$
$$=(\bar{x}_m+1)^{n-p+q+1}$$

So $t-m-1<n-p+q+1$. This leads to $m\geq p-n$.

Let $M=\min(t,p-q)$. We now show $\mu=\sigma(\bar{\pi}')$, $\sigma(\bar{\pi}')-\sigma(\bar{\pi})=$

$$\frac{\bar{x}_t-\bar{x}_m-1}{\bar{x}_m\bar{x}_t}\sum_{t-q}^{m}\bar{x}_i\cdots\bar{x}_{i+q}+\frac{1}{\bar{x}_m}\sum_{m-q}^{t-q-1}\bar{x}_i\cdots\bar{x}_{i+q}-\frac{1}{\bar{x}_t}\sum_{m+1}^{t}\bar{x}_i\cdots\bar{x}_{i+q}$$

$$=\frac{1}{\bar{x}_m\bar{x}_t}\sum_{t-q}^{m}\bar{x}_i\cdots\bar{x}_{i+q}-\frac{1}{\bar{x}_t}\sum_{m+1}^{M}\bar{x}_i\cdots\bar{x}_{i+q}$$

$$\geq\sum_{i=t}^{n}\frac{\bar{x}_m\cdots\bar{x}_i}{\bar{x}_m\bar{x}_t}\cdot\frac{\bar{x}_{i+1}\cdots\bar{x}_n}{\bar{x}_{m+1}\cdots\bar{x}_{n+m-i}}-\sum_{i=m+1}^{M}\frac{\bar{x}_i\cdots\bar{x}_n}{\bar{x}_t}$$

$$=\sum_{i=t}^{n}\frac{\bar{x}_{n-m-i+1}\cdots\bar{x}_n}{\bar{x}_t}-\sum_{i=m+1}^{M}\frac{\bar{x}_i\cdots\bar{x}_n}{\bar{x}_t}$$

$$=\sum_{i=m+1}^{n+m-t+1}\frac{\bar{x}_i\cdots\bar{x}_n}{\bar{x}_t}-\sum_{i=m+1}^{M}\frac{\bar{x}_i\cdots\bar{x}_n}{\bar{x}_t}$$

$$\geq\sum_{i=m+1}^{M}\frac{\bar{x}_i\cdots\bar{x}_n}{\bar{x}_t}-\sum_{i=m+1}^{M}\frac{\bar{x}_i\cdots\bar{x}_n}{\bar{x}_t}$$

$$=0.$$

Hence $\mu=\sigma(\bar{\pi}')$.

In the third step, multiplication by $\dfrac{\bar{x}_{i+1}\cdots\bar{x}_n}{\bar{x}_{m+1}\cdots\bar{x}_{n+m-i}}$ is permissible because $\dfrac{\bar{x}_{i+1}\cdots\bar{x}_n}{\bar{x}_{m+1}\cdots\bar{x}_{n+m-i}} \leq \dfrac{(\bar{x}_m+1)^{n-i}}{(\bar{x}_m+2)^{n-i}} \leq 1.$ Use of $m\geq p-n$ is made in the sixth step to prove $n+m-t+1 \geq p-q \geq M.$ ∎

For reference through theorem 1.2.22, we state some conditions for partitions in Π.

(a) $\mu=\sigma(\pi)$.

(b) $x_i=1$ for $i=1,\ldots,m-1,n+1,\ldots,p$, where $n-m=q$.

(c) If $x_i\neq x_j$ for some i and j, $m\leq i,j\leq n$, then there are numbers r and s, $m\leq r\leq s\leq n$, such that $x_i=A$ for $i=m,\ldots,r-1,s+1,\ldots,n$, and $x_i=A+1$ for $i=r,\ldots,s$.

(d) $2\leq A$.

The number A in (c) and (d) is defined to be $A=\min(x_m,x_n)$. A partition satisfying (a)-(d) is denoted $\bar{\pi}$. We continue to defer the case where some of $x_m,\ldots,x_n$ are one. The final goal before computing μ is to prove stringent restrictions on the positions of m,n,r, and s. We accomplish this in the next two theorems.

Lemma 1.2.20. In $\bar{\pi}$ one of the following is true:

(i) $m\leq 1+q$ and $n\geq p-q$.

(ii) $m\geq 1+q$ and $n\leq p-q$.

Proof If $p\leq 1+2q$, then $m\leq p-q\leq 1+q\leq n$ and (i) is true. So assume without loss of generality that $p>1+2q$.

Let π satisfy (b)-(d), but assume $m<1+q$ and $n<p-q$. We show $\mu\neq\sigma(\pi)$.

By lemma 1.2.4,

$$\begin{aligned}\sigma(\vec{\pi})-\sigma(\pi)&=x_p x_1\ldots x_q-x_{p-q}\ldots x_p\\&\geq x_m-1\\&>0.\end{aligned}$$

Hence $\mu\neq\sigma(\pi)$.

Thus in $\bar{\pi}$, if $m<1+q$, then $n\geq p-q$, i.e. (i) is true. A similar argument establishes that in $\bar{\pi}$, if $m>1+q$ then (ii) is satisfied. ∎

Theorem 1.2.21 If in $\bar{\pi}$, $\bar{x}_i=\bar{x}_m$ for $i=m,\ldots,n$, then there is a $\bar{\pi}'$ such that in $\bar{\pi}'$, $(m+n)-(1+p)=0$ or 1.

Proof.

Case 1 $m\leq 1+q$ and $n\geq p-q$. Assume without loss of generality that $p>1+q$ and that $m>1$. Let π satisfy (b)-(d), but assume that $(m+n)-(1+p)\geq 2$. We show $\mu\neq\sigma(\pi)$.

From lemma 1.2.4,

$$\sigma(\vec{\pi})-\sigma(\pi)=x_p x_1\ldots x_q-x_{p-q}\ldots x_p$$
$$\geq A^{1+q-m}-A^{n-(p-q)+1}$$
$$>0.$$

Hence $\mu\neq\sigma(\pi)$.

So in π, $(m+n)-(1+p)\leq 1$. A similar argument establishes that $(m+n)-(1+p)\geq -1$. So the conclusion holds in either π or $\hat{\pi}$.

Case 2 $m\geq 1+q$ and $n\leq p-q$. In this case both $\sigma(\vec{\pi})-\sigma(\pi)$ and $\sigma(\vec{\pi})-\sigma(\pi)$ vanish. So the sequence $x_m,\ldots,x_n$ can be shifted to the required position to produce $\bar{\pi}'$ in which $\mu=\sigma(\pi')$.

Lemma 1.2.20 guarantees that the two cases are exhaustive. ∎

Theorem 1.2.22 There exists a $\bar{\pi}'$ satisfying

(i) $(m+n)-(1+p)=0$ or 1

(ii) $(r+s)-(1+p)=0$ or 1

Proof. We consider two main cases: $1+q\leq m$, $n\leq p-q$ and $m\leq 1+q$, $p-q\leq n$. These are exhaustive in light of lemma 1.2.20.

Case 1 $1+q\leq m$, $n\leq p-q$. We establish (i) by the same argument as in case 2, theorem 1.2.21.

Assume $(r+s)-(1+p)\geq 2$. Now from lemma 1.2.1,

$$\sigma(\bar{\pi}<r-1,s>)-\sigma(\bar{\pi})=\frac{1}{\bar{x}_{r-1}}\sum_{r-1-q}^{s-q-1}\bar{x}_i\ldots\bar{x}_{i+q}-\frac{1}{\bar{x}_s}\sum_{i=r}^{s}\bar{x}_i\ldots\bar{x}_{i+q}$$

$$=\frac{\bar{x}_m\ldots\bar{x}_{r-1}}{\bar{x}_{r-1}}\sum_{i=r-1}^{s-1}\frac{\bar{x}_{r-1}\ldots\bar{x}_i}{\bar{x}_{r-1}}-\frac{\bar{x}_s\ldots\bar{x}_n}{\bar{x}_s}\sum_{i=r}^{s}\frac{\bar{x}_i\ldots\bar{x}_s}{\bar{x}_s}$$

$$=(A^{r-1-m}-A^{n-s})\sum_{i=r}^{s}\frac{\bar{x}_i\ldots\bar{x}_s}{\bar{x}_s}.$$

$$\geq 0$$

$=0$ only if $(m+n)-(1+p)=1$ and $(r+s)-(1+p)=2$.

Therefore, by selecting $\bar{\pi}'$ to be $\bar{\pi}$ or, if necessary, to be $\bar{\pi}\langle r-1,s\rangle$, we have (i) and (ii). If $(r+s)-(1+p)\leq -1$, a similar argument proves the same result.

Case 2 $m\leq 1+q$, $p-q\leq n$. If $p=1+q$, an appropriate permutation makes the theorem trivial.

Assume now $p=2+q$. Without loss of generality assume $m=2$. Then $(m+n)-(1+p)=1$ and we have (i).

We show $\bar{x}_n\leq\bar{x}_2$. For if $\bar{x}_n>\bar{x}_2$ then by lemma 1.2.4,

$$\begin{aligned}\sigma(\overleftarrow{\pi})-\sigma(\pi)&=-\bar{x}_1\cdots\bar{x}_{1+q}+\bar{x}_{p-q+1}\cdots\bar{x}_p\bar{x}_1\\ &=\bar{x}_m\cdots\bar{x}_n\left(\frac{1}{\bar{x}_2}-\frac{1}{\bar{x}_n}\right)\\ &>0.\end{aligned}$$

This contradiction proves that $\bar{x}_n\leq\bar{x}_2$ in $\bar{\pi}$.

If therefore $s=n$ then $r=2$, and hence $(r+s)-(1+p)=1$. If $s<n$, an appropriate permutation of $\bar{x}_2,\ldots,\bar{x}_{p-1}$ will produce $\bar{\pi}'$ satisfying (i) and (ii).

Assume henceforth that $p\geq 3+q$. We show first that $n<p$. For if $n=p$, then by lemma 1.2.4,

$$\begin{aligned}\sigma(\overleftarrow{\pi})-\sigma(\bar{\pi})&=-\bar{x}_1\cdots\bar{x}_{1+q}+\bar{x}_{p-q+1}\cdots\bar{x}_p\bar{x}_1\\ &\geq\bar{x}_{m+1}\cdots\bar{x}_n-\bar{x}_m\cdots\bar{x}_{n-2}\\ &=\bar{x}_m\cdots\bar{x}_n\left(\frac{1}{\bar{x}_m}-\frac{1}{\bar{x}_{n-1}\bar{x}_n}\right)\\ &>0.\end{aligned}$$

This contradiction proves $n<p$. Similarly $m>1$.

We note that in $\bar{\pi}$ the following inequalities hold:

$$(*)\qquad\begin{aligned}&\bar{x}_{p-q+1}\cdots\bar{x}_n\leq\bar{x}_m\cdots\bar{x}_{1+q}.\\ &\bar{x}_m\cdots\bar{x}_q\leq\bar{x}_{p-q}\cdots\bar{x}_n.\end{aligned}$$

Indeed, if either failed, lemma 1.2.4 and $\bar{x}_1=\bar{x}_p=1$ would give an immediate contradiction.

Case 2a $p<1+2q$ From lemma 1.2.14 we have that one of the following is true.

$p-q\le r\le s\le 1+q$.

$r\le p-q\le 1+q\le s$.

Dividing the equations (*) by $\bar{x}_{p-q+1}\cdots\bar{x}_{1+q}$ and $\bar{x}_{p-q}\cdots\bar{x}_{q}$ yields

$$(**)\quad \begin{array}{l}\bar{x}_{2+q}\cdots\bar{x}_{n}\le\bar{x}_{m}\cdots\bar{x}_{p-q}.\\ \bar{x}_{m}\cdots\bar{x}_{p-q-1}\le\bar{x}_{1+q}\cdots\bar{x}_{n}.\end{array}$$

We subdivide case 2a into two more cases.

Case $2a_1$ $p-q\le r\le s\le 1+q$. From the first equation in (**) we get

$A^{n-(2+q)+1}\le A^{p-q-m}(A+1)$. Therefore,

$$\begin{aligned} n-(2+q)+1 &\le p-q-m+\log_A (A+1)\\ &\le p-q-m+1\end{aligned}$$

Consequently $(n+m)\le(1+p)+1$.

In a similar manner, the second equation in (**) gives us $(1+p)\le(n+m)+1$. Therefore in $\bar{\pi}$, $|(m+n)-(1+p)|\le 1$. So (i) is true in either $\bar{\pi}$ or $\hat{\bar{\pi}}$. An appropriate permutation of $\bar{x}_{p-q},\ldots,\bar{x}_{1+q}$ gives (ii) immediately.

Case $2a_2$ $r\le p-q\le 1+q\le s$. We first prove (ii). Suppose $(r+s)\ge(p+1)+2$. Then by lemma 1.2.1, $\sigma(\bar{\pi}<r-1,s>)-\sigma(\bar{\pi})=$

$$\frac{1}{\bar{x}_{r-1}}\sum_{r-1-q}^{s-q-1}\bar{x}_{i}\cdots\bar{x}_{i+q}-\frac{1}{\bar{x}_{s}}\sum_{r}^{s}\bar{x}_{i}\cdots\bar{x}_{i+q}$$

$$=\frac{\bar{x}_{m}\cdots\bar{x}_{1+q}}{\bar{x}_{r-1}}\sum_{i=1+q}^{s-1}\frac{\bar{x}_{1+q}\cdots\bar{x}_{i}}{\bar{x}_{1+q}}-\frac{\bar{x}_{p-q}\cdots\bar{x}_{n}}{\bar{x}_{s}}\sum_{i=r}^{p-q}\frac{\bar{x}_{i}\cdots\bar{x}_{p-q}}{\bar{x}_{p-q}}$$

$$\geq \bar{x}_{p-q+1}\cdots\bar{x}_n\left(\frac{1}{\bar{x}_m}\sum_{i=1+q}^{s-1}\frac{\bar{x}_{1+q}\cdots\bar{x}_i}{\bar{x}_{1+q}} - \sum_{i=r}^{p-q}\frac{\bar{x}_i\cdots\bar{x}_{p-q}}{\bar{x}_{p-q}}\right)$$

$$=\bar{x}_{p-q+1}\cdots\bar{x}_n\left(\frac{1}{\bar{x}_m}\quad\frac{(\bar{x}_m+1)^{s-q}-1}{\bar{x}_m} - \frac{(\bar{x}_m+1)^{p-q-r+2}-1}{\bar{x}_m}\right)$$

$$\geq\frac{\bar{x}_{p-q+1}\cdots\bar{x}_n}{\bar{x}_m^{\ 2}}\left((\bar{x}_m+1)^{s-q}-1-(\bar{x}_m+1)^{p-q-r+3}+\bar{x}_m\right)$$

$$\geq\frac{\bar{x}_{p-q+1}\cdots\bar{x}_n}{\bar{x}_m^{\ 2}}\left((\bar{x}_m+1)^{p-q-r+3}-1-(\bar{x}_m+1)^{p-q-r+3}+\bar{x}_m\right)$$

$$>0.$$

This contradiction proves $r+s\leq(p+1)+1$. A similar dual argument yields $(p+1)\leq(r+s)+1$. Thus $|(r+s)-(1+p)|\leq 1$. Therefore (ii) holds in either $\bar{\pi}$ or $\hat{\bar{\pi}}$.

Now using (*) we get,

$$\bar{x}_{p-q+1}\cdots\bar{x}_n \leq \bar{x}_m\cdots\bar{x}_{1+q}.$$

$$\bar{x}_{1+q}\cdots\bar{x}_n \leq \bar{x}_m\cdots\bar{x}_{p-q}\cdot\bar{x}_{1+q}.$$

$$(\bar{x}_m+1)^{r+s-(1+p)}\bar{x}_m^{\ n-s} \leq \bar{x}_m^{\ r-m}(\bar{x}_m+1).$$

Hence, $$(n+m)-(r+s) \leq\begin{cases}0 \text{ if } (r+s)-(1+p)=1\\ 1 \text{ if } (r+s)-(1+p)=0\end{cases}$$

Similarly, using the second equation of (*) we get

$$(n+m)-(r+s) \geq\begin{cases}-1 \text{ if } (r+s)-(p+1)=0\\ -2 \text{ if } (r+s)-(p+1)=1\end{cases}$$

These results together give us $|(n+m)-(p+1)|\leq 1$. We next show that if $(n+m)-(p+1)=-1$, then $(r+s)-(p+1)=0$. For assume that $(n+m)-(p+1)=-1$ and $(r+s)-(p+1)=1$. Then by lemma 1.2.1,

$$\sigma(\pi<r-1,s>)-\sigma(\pi)=\frac{\bar{x}_m\cdots\bar{x}_{1+q}}{\bar{x}_{r-1}}\sum_{i=1+q}^{s-1}\frac{\bar{x}_{1+q}\cdots\bar{x}_i}{\bar{x}_{1+q}} -$$

$$- \frac{\bar{x}_{p-q}\cdots\bar{x}_n}{\bar{x}_s} \sum_{i=r}^{p-q} \frac{\bar{x}_i\cdots\bar{x}_{p-q}}{\bar{x}_{p-q}}$$

$$= \bar{x}_m^{\,r-m-1}(\bar{x}_m+1)^{1+q-r+1}\,\frac{(\bar{x}_m+1)^{s-q-1}-1}{\bar{x}_m}$$

$$- \bar{x}_m^{\,n-s}(\bar{x}_m+1)^{s-p+q}\cdot\frac{(\bar{x}_m+1)^{p-q-r+1}-1}{\bar{x}_m}$$

$$= \frac{(\bar{x}_m+1)^{2+q-r}\bar{x}_m^{\,r-m-2}(\bar{x}_m+1)^{s-q-1}}{\bar{x}_m}(\bar{x}_m-1)$$

$$> 0.$$

This contradiction proves that if $(n+m)-(p+1)=-1$, then $(r+s)-(p+1)=0$.

If now $(n+m)-(p+1)=-1$, then $\hat{\bar{\pi}}$ satisfies (i) and (ii).

<u>Case 2b $p\geq 1+2q$</u>. We first show that $\bar{\pi}$ can be selected so that one of the following is true:

$1+q\leq r\leq s\leq p-q$.

$r\leq 1+q\leq p-q\leq s$.

If in $\bar{\pi}$ $2+q<r\leq p-q\leq s$, then by lemma 1.2.1

$$\sigma(\bar{\pi}<r-1,s>)-\sigma(\bar{\pi})=\frac{1}{\bar{x}_{r-1}}\sum_{r-1-q}^{s-q-1}\bar{x}_i\cdots\bar{x}_{i+q}-\frac{1}{\bar{x}_s}\sum_{r}^{s}\bar{x}_i\cdots\bar{x}_{i+q}$$

$$=\frac{\bar{x}_m\cdots\bar{x}_{r-1}}{\bar{x}_{r-1}}\sum_{i=r-1}^{s-1}\frac{\bar{x}_{r-1}\cdots\bar{x}_i}{\bar{x}_{r-1}}-\frac{\bar{x}_{p-q}\cdots\bar{x}_n}{\bar{x}_s}\sum_{i=r}^{p-q}\frac{\bar{x}_i\cdots\bar{x}_{p-q}}{\bar{x}_{p-q}}$$

$$\geq\bar{x}_m\cdots\bar{x}_{1+q}\,\frac{(\bar{x}_m+1)^{s-r+1}-1}{\bar{x}_m}-\bar{x}_{p-q+1}\cdots\bar{x}_n\,\frac{(\bar{x}_m+1)^{p-q-r+1}-1}{\bar{x}_m}$$

$$\geq\frac{\bar{x}_{p-q+1}\cdots\bar{x}_n}{\bar{x}_m}\left((\bar{x}_m+1)^{s-r+1}-(\bar{x}_m+1)^{p-q-r+1}\right)$$

$$\geq 0.$$

So $\mu=\sigma(\bar{\pi}')$.

Actually, the only change in $\bar{\pi}<r-1,s>$ is that the sequence $\bar{x}_r,\ldots,\bar{x}_s$ has been shifted to the left one position. A sufficient number of such shifts will result in either $1+q\le r\le s\le p-q$ or $r\le 1+q\le p-q\le s$.

If $r\le 1+q\le s<p-q-1$ a dual argument proves the same result.

<u>Case $2b_1$ $1+q\le r\le s\le p-q$.</u>

Equations (*) become

$$A^{n-(p-q+1)+1} \le A^{1+q-m}(A+1),$$
$$A^{q-m+1} \le A^{n-(p-q)}(A+1).$$

These give us

$m+n \le (p+1)+1,$

$(p+1)\le(n+m)+1.$

Hence $|(n+m)-(p+1)|\le 1$. In one of $\bar{\pi}$ or $\hat{\pi}$, (i) is now true.

We show next that $|(r+s)-(m+n)|\le 1$. For assume $(r+s)-(m+n)\ge 2$.

Then $\sigma(\bar{\pi}<r-1,s>)-\sigma(\bar{\pi})=$

$$\frac{1}{\bar{x}_{r-1}}\sum_{r-1-q}^{s-q-1}\bar{x}_i\cdots\bar{x}_{i+q} - \frac{1}{\bar{x}_s}\sum_{r}^{s}\bar{x}_i\cdots\bar{x}_{i+q}$$

$$=\frac{\bar{x}_m\cdots\bar{x}_{r-1}}{\bar{x}_{r-1}}\sum_{i=r-1}^{s-1}\frac{\bar{x}_{r-1}\cdots\bar{x}_i}{\bar{x}_{r-1}} - \frac{\bar{x}_s\cdots\bar{x}_n}{\bar{x}_s}\sum_{i=r}^{s}\frac{\bar{x}_i\cdots\bar{x}_s}{\bar{x}_s}$$

$$=(A^{r-m-1}-A^{n-s})\sum_{i=r}^{s}\frac{\bar{x}_i\cdots\bar{x}_s}{\bar{x}_s}$$

$$>0.$$

This contradiction establishes that $(r+s)-(m+n)\le 1$. A similar argument gives $(m+n)-(r+s)\le 1$. Hence $|(r+s)-(m+n)|\le 1$.

Using the comparisons in the preceding paragraph, we see that

if $(r+s)-(m+n)=1$, then $\mu=\sigma(\bar{\pi})=\mu(\bar{\pi}<r-1,s>)$;

if $(r+s)-(m+n)=-1$, then $\mu=\sigma(\bar{\pi})=\sigma(\bar{\pi}<s+1,r>)$.

So now if $(r+s)-(m+n)=0$, let $\bar{\pi}'=\bar{\pi}$.

If $(r+s)-(m+n)=-1$, let $\bar{\pi}'=\begin{cases}\bar{\pi} & \text{if } (m+n)-(1+p)=1\\ \bar{\pi}<s+1,r> & \text{if } (m+n)-(1+p)=0\end{cases}$

If $(r+s)-(m+n)=1$, let $\bar{\pi}'=\begin{cases}\bar{\pi}<r-1,s> & \text{if } (m+n)-(1+p)=1\\ \bar{\pi} & \text{if } (m+n)-(1+p)=0\end{cases}$

With the above choices for $\bar{\pi}'$, condition (ii) is true.

<u>Case $2b_2$ $r\leq 1+q\leq p-q\leq s$</u>. We first show $|(r+s)-(p+1)|\leq 1$. For assume $(r+s)-(p+1)\geq 2$. Then by lemma 1.2.1, $\sigma(\bar{\pi}<r-1,s>)-\sigma(\bar{\pi})=$

$$=\frac{1}{\bar{x}_{r-1}}\sum_{r-1-q}^{s-1-q}\bar{x}_i\cdots\bar{x}_{i+q}-\frac{1}{\bar{x}_s}\sum_{r}^{s}\bar{x}_i\cdots\bar{x}_{i+q}$$

$$\geq\bar{x}_{p-q+1}\cdots\bar{x}_n\left(\frac{1}{A}\sum_{i=1+q}^{s-1}\frac{\bar{x}_{1+q}\cdots\bar{x}_i}{\bar{x}_{1+q}}-\sum_{i=r}^{p-q}\frac{\bar{x}_i\cdots\bar{x}_{p-q}}{\bar{x}_{p-q}}\right)$$

$$=\bar{x}_{p-q+1}\cdots\bar{x}_n\left(\frac{(A+1)^{s-q-1}-1}{A^2}-\frac{(A+1)^{p-q-r+1}-1}{A}\right)$$

$$=A^{-2}\bar{x}_{p-q+1}\cdots\bar{x}_n\left((A+1)^{s-q-1}-1-A(A+1)^{p-q-r+1}+A\right)$$

$$>A^{-2}\bar{x}_{p-q+1}\cdots\bar{x}_n\left((A+1)^{s-q-1}-(A+1)^{p-q-r+2}\right)$$

$$\geq 0.$$

This contradiction proves that $(r+s)-(p+1)\leq 1$. A similar argument proves $(p+1)-(r+s)\leq 1$. Hence in one of $\bar{\pi}$ or $\hat{\bar{\pi}}$, (ii) is satisfied.

The first equation in (*) becomes on substitution $A^{n-s}(A+1)^{s-p+q}\leq A^{r-m}(A+1)^{2+q-r}$. From this we get

$(m+n)-(r+s)\leq 1$ if $(r+s)-(p+1)=0$;

$(m+n)-(r+s)\leq 0$ if $(r+s)-(p+1)=1$.

Similarly, the second equation in (*) yields

$(m+n)-(r+s)\geq -1$ if $(r+s)-(p+1)=0$

$(m+n)-(r+s)\geq -3$ if $(r+s)-(p+1)=1$.

If $(r+s)-(p+1)=0$, then (i) and (ii) both hold in one of $\bar{\pi}$ or $\hat{\bar{\pi}}$.

If $(r+s)-(p+1)=1$, we show $(m+n)-(r+s)\geq -1$. For assume $(m+n)-(r+s)\leq -2$. Then $\sigma(\bar{\pi}<r-1,s>)-\sigma(\bar{\pi})=$

$$\frac{\bar{x}_m\cdots\bar{x}_{1+q}}{\bar{x}_{r-1}}\sum_{i=1+q}^{s-1}\frac{\bar{x}_{1+q}\cdots\bar{x}_i}{\bar{x}_{1+q}}-\frac{\bar{x}_{p-q}\cdots\bar{x}_n}{\bar{x}_s}\sum_{i=r}^{p-q}\frac{\bar{x}_i\cdots\bar{x}_{p-q}}{\bar{x}_{p-q}}$$

$$=A^{r-m-1}(A+1)^{2+q-r}\,\frac{(A+1)^{s-q-1}-1}{A}$$

$$-A^{n-s}(A+1)^{s-p+q-1}\,\frac{(A+1)^{p-q-r+1}-1}{A}$$

$$=A^{n-s-1}(A+1)^{s-p+q-1}\Big(A(A+1)\big((A+1)^{s-q-1}-1\big)-(A+1)^{p-q-r+1}-1\Big)$$

$$=A^{n-s-1}(A+1)^{s-p+q-1}(A(A+1)-1)\big((A+1)^{p-q-r-1}-1\big)$$

$$\geq A^{n-s-1}(A+1)^{s-p+q-1}(A(A+1)-1)A$$

$$>0.$$

This contradiction proves that $(m+n)-(r+s)\geq -1$.

Thus if $(r+s)-(p+1)=1$, then $(m+n)-(r+s)=0$ or 1. So (i) is true. ∎

We now show that our results are valid even if some of $x_m,\ldots,x_n$ are one. Evidently in this case we have some freedom in the choice of m and n. Lemmas 1.1.11-13 allow us to assume $x_1\leq\cdots\leq x_{1+q}$. We now select n so that $x_n\geq 2$ and $x_i=1$ for $i=n+1,\ldots,p$. Then m is chosen to be $n-q$. Because $n\geq 1+q$, we have $m\geq 1$. With this choice of m and n

lemmas 1.2.17, 1.2.18, and theorem 1.2.19 remain valid. It is then straightforward to verify that in theorem 1.2.22 a $\overline{\pi}'$ can be found satisfying (ii) and satisfying (i) for some new choice of m and n. ▮

In theorems 1.2.2. and 1.2.22 there is exactly one partition satisfying the conclusions. We give an explicit description of this partition in terms of N,p, and q.

Theorem 1.2.22 With π as defined below, $\mu=\sigma(\pi)$.

Let $A=\dfrac{N-p+q+1}{q+1}$.

Define m,n,r, and s by

$m=\left[\dfrac{p-q}{2}\right]+1,$

$n=\left[\dfrac{p+q}{2}\right]+1,$

$r=\left[\dfrac{p-(q+1)(A-[A])}{2}\right]+1,$

$s=\left[\dfrac{p+(q+1)(A-[A])}{2}\right].$

Now define x_i by

$x_i=1$ for $i=1,\ldots,m-1,n+1,\ldots,p$

$x_i=[A]$ for $i=m,\ldots,r-1,s+1,\ldots,m$

$x_i=[A]+1$ for $i=r,\ldots,s$.

Proof. It is routine to check that m,n,r, and s satisfy the conditions in theorems 1.2.21 and 1.2.22, and that

$\sum_{i=1}^{p} x_i=N.$ ▮

Using the partition in theorem 1.2.22 it is possible to compute μ exactly. The general formula is too ponderous to be interesting. We give instead a simpler approximation for μ which does not involve r and s.

Theorem 1.2.23

If $p \geq 1+3q$,

$$N-3q-1+\frac{[A]^{q+1}+[A]^{q}-2[A]}{[A]-1} \leq \mu < N-3q-1+\frac{([A]+1)^{q+1}+([A]+1)^{q}-2([A]+1)}{[A]}.$$

If $p<1+3q$,

$$2\frac{[A]^{q+1}-[A]^{2+q-m}}{[A]-1}+[A]^{q+1} \leq \mu < 2\frac{([A]+1)^{q+1}-([A]+1)^{n-p+q+1}}{[A]}+([A]+1)^{q+1}.$$

In either case, if $[A]=1$, then an alternate lower bound for μ is $p-q \leq \mu$.

Proof. The lower bounds are found when all of $x_m,\ldots,x_n$ are $[A]$. The upper bounds are found when all the terms $x_m,\ldots,x_n$ are $[A]+1$.

Chapter II

Maximizing over Nonnegative Real Numbers

We turn to the second problem stated in chapter 0. The three fixed parameters are α, p, and q, where α is a positive real number and $1 \leq q \leq p-1$. The variables x_i are now non-negative real numbers. If E^p denotes euclidean p-space, then Π is given by

$$\Pi=\{\pi=(x_1,\ldots,x_p)\varepsilon E^p \mid x_i \geq 0 \quad \text{for } i=1,\ldots,p; \quad \sum_{i=1}^{p} x_i=\alpha\}.$$

Let $R_{\geq 0}$ denote the set of nonnegative real numbers. Then the mapping $\sigma:\Pi \longrightarrow R_{\geq 0}$ and the number μ are defined analogously to the first problem, i.e. $\sigma(\pi)=\sum_{i=1}^{p-q} x_i\ldots x_{i+q}$ and $\mu=\max_{\pi\varepsilon\Pi} \sigma(\pi)$.

In theorem 2.1 we prove that $\mu=\sigma(\bar{\pi})$ for some $\bar{\pi}\varepsilon\Pi$. The results of this chapter are stated in theorem 2.5.

<u>Theorem 2.1</u> The number μ exists and there exists a partition $\bar{\pi}\varepsilon\Pi$ such that $\mu=\sigma(\bar{\pi})$.

<u>Proof</u>. We show that Π is a compact subset of E^p and that $\sigma:\Pi\rightarrow R_{\geq 0}$ is continuous.

Let d denote the standard metric for E^p, and let $\theta=(0,\ldots,0)$ denote the zero element in E^p. For $\pi\varepsilon\Pi$, $0\leq x_i \leq \alpha$ for $i=1,\ldots,p$. Therefore

$$d(\theta,\pi)=\sqrt{\sum_1^p (x_i-0)^2}$$

$$<\sqrt{p\alpha^2}$$

$$<p\alpha.$$

So for $\pi,\pi'\varepsilon\Pi$,

$d(\pi,\pi')\leq d(\theta,\pi)+d(\theta,\pi')<2p\alpha$.

Hence the set Π is bounded.

Let $(\pi^j)_{j=1}^{\infty}$ be a Cauchy sequence of elements in Π with limit π in E^p. We show $\pi\varepsilon\Pi$. Convergence in E^k is equivalent to convergence in every coordinate. For i fixed,

$x_i = \lim_{j\to\infty} x_i^j$

≥ 0, because $x_i^j \geq 0$ for all j.

That is, in π $x_i \geq 0$ for $i=1,\ldots,p$.

Now because $(\pi^j)_{j=1}^{\infty}$ is Cauchy in each coordinate, for a fixed $\varepsilon>0$ and for a fixed i, there exists an

$N(\varepsilon,i)$ such that for all $j\geq N(\varepsilon,i)$, $|x_i^j-x_i|<\varepsilon$.

Let $N(\varepsilon)=\max_{i=1}^{p} N(\varepsilon,i)$. Then for a fixed $\varepsilon>0$, there exists an

$N(\varepsilon)$ such that for all $j\geq N(\varepsilon)$ and for all i, $|x_i^j-x_i|<\varepsilon$.

Now let $\varepsilon>0$ be arbitrary but fixed. Select N so large that for all $j\geq N$ and for all i, $|x_i^j-x_i|<\frac{\varepsilon}{p}$. Then for all $j\geq N$,

$$\left|\sum_{i=1}^{p} x_i-\alpha\right|=\left|\sum_{i=1}^{p} x_i - \sum_{i=1}^{p} x_i^j\right|$$

$$=\left|\sum_{i=1}^{p}(x_i-x_i^j)\right|$$

$$\leq \sum_{i=1}^{p}\left|x_i-x_i^j\right|$$

$$< k\cdot\frac{\varepsilon}{k}$$

$$<\varepsilon.$$

Hence $\sum_{i=1}^{p} x_i=\alpha$. Combined with the result that $x_i\geq 0$ for $i=1,\ldots,p$, this proves that $\pi\varepsilon\Pi$.

Thus Π contains all its limit points. So Π is a closed subset of E^p. It was shown earlier that Π is bounded. Consequently Π is compact in E^p.

The mapping $\sigma:\Pi \rightarrow R_{\geq 0}$ consists of sums of products of projections, which are continuous. So σ is continuous.

So, using a well-known theorem from topology, it must be true that σ attains a maximum μ for some $\bar{\pi}\varepsilon\Pi$. ▮

<u>Lemma 2.2</u> Let the distinct integers h and k in the range $1 \leq h,k \leq p$ be such that $|h-k|>q$. Let $\pi=(x_1,\ldots,x_p)\varepsilon\Pi$ be such that $x_k=\varepsilon>0$. Then

$$\sigma(\pi<h,k>)-\sigma(\pi)=\frac{x_k}{x_h}\sum_{i=h-q}^{h} x_i\ldots x_{i+q} - \sum_{i=k-q}^{k} x_i\ldots x_{i+q}.$$

<u>Proof</u>. We omit the proof, for it involves the same technique as lemma 1.2.1. ▮

<u>Theorem 2.3</u> There exists a partition $\bar{\pi}^o$ such that $\bar{x}_i^o=0$ for $i=2+q,\ldots,p$.

<u>Proof</u>. We first prove the following: if $1 \leq h \leq k-q-1$, $k \leq p$, then there exists a partition $\bar{\pi}$ such that $\bar{x}_i=0$ for $i=1,2,\ldots,h-1,k+1,\ldots,p$, and where $x_h=0$ or $x_k=0$. The proof is by induction on k-h from p-1 to q+1.

To start the induction, let k-h=p-1. Then k=p and h=1. The condition $x_i=0$ for $i=1,2,\ldots,j-1,k+1,\ldots,p$ is true by convention. Assume without loss of generality that $\bar{x}_h,\bar{x}_k>0$.

Define π' and π'' by

$\pi'=\bar{\pi}<1,p>$ where $\varepsilon=x_p$, and

$\pi''=\bar{\pi}<p,1>$ where $\varepsilon=x_1$.

Then by lemma 2.2,

$$\sigma(\pi')-\sigma(\bar{\pi})=\frac{\bar{x}_p}{\bar{x}_1}\sum_{1-q}^{1}\bar{x}_i\ldots\bar{x}_{i+q}-\sum_{p-q}^{p}\bar{x}_i\ldots\bar{x}_{i+q}$$

$$=\bar{x}_p\left(\frac{1}{\bar{x}_1}\sum_{1-q}^{1}\bar{x}_i\ldots\bar{x}_{i+q}-\frac{1}{x_p}\sum_{p-q}^{p}\bar{x}_i\ldots\bar{x}_{i+q}\right)$$

$$\sigma(\pi'')-\sigma(\bar{\pi})=\sum_{1-q}^{1}\bar{x}_i\ldots\bar{x}_{i+q}+\frac{\bar{x}_1}{\bar{x}_p}\sum_{p-q}^{p}\bar{x}_i\ldots\bar{x}_{i+q}$$

$$=\bar{x}_1\left(-\frac{1}{\bar{x}_1}\sum_{1-q}^{1}\bar{x}_i\ldots\bar{x}_{i+q}+\frac{1}{\bar{x}_p}\sum_{p-q}^{p}\bar{x}_i\ldots\bar{x}_{i+q}\right).$$

It follows that $\sigma(\pi')-\sigma(\bar{\pi})\geq 0$ or $\sigma(\pi'')-\sigma(\bar{\pi})\geq 0$

If $\sigma(\pi')-\sigma(\bar{\pi})\geq 0$, then $\mu=\sigma(\bar{\pi})=\sigma(\pi')$ and $x_p=0$.

If $\sigma(\pi'')-\sigma(\bar{\pi})\geq 0$, then $\mu=\sigma(\bar{\pi})=\sigma(\pi'')$ and $x_1=0$.

In either case the desired partition has been produced, proving the statement for $k-h=p-1$.

Now let h and k be such that $1\leq h$, $k\leq p$, and $q+1\leq k-h\leq p-2$. Without loss of generality assume $x_h, x_k>0$.

Case 1 $h>2$.

Applying the inductive hypothesis to h-1 and k yields a partition $\bar{\pi}$ such that $\bar{x}_i=0$ for $i=1,2,\ldots,h-2,k+1,\ldots,p$, and such that $\bar{x}_{h-1}=0$.

Define π' and π'' by

$\pi'=\bar{\pi}\langle h,k\rangle$ for $\varepsilon=\bar{x}_k$, and

$\pi''=\bar{\pi}\langle k,h\rangle$ for $\varepsilon=\bar{x}_h$.

As above, $\sigma(\pi'')-\sigma(\bar{\pi})\geq 0$ or $\sigma(\pi'')-\sigma(\bar{\pi})\geq 0$. If $\sigma(\pi'')-\sigma(\bar{\pi})\geq 0$, then $\sigma(\pi'')=(\bar{\pi})=\mu$ and $\bar{x}_k=0$. If $\sigma(\pi')-\sigma(\bar{\pi})\geq 0$, then $\sigma(\pi')=\sigma(\bar{\pi})=\mu$ and $\bar{x}_h=0$. In either case the desired partition has been produced. Because $(k-h)<(k-(h-1))$, the statement is proved by induction.

Case 2 $h=1$ In this case the inductive hypothesis must be applied to h and k+1. A similar argument works thereafter.

In particular, for $h=k-q-1$, there is a partition $\bar{\pi}$ such that $\bar{x}_i=0$ for $i=1,\ldots,h-1,k+1,\ldots,p$, and $\bar{x}_h=0$ or $\bar{x}_k=0$. Or, with appropriate relabling, there is a partition $\bar{\pi}$ such that $\bar{x}_i=0$ for $i=1,\ldots,m-1,n+1,\ldots,p$, where m and n are such that $1\leq m,n\leq p, n-m=q$.

Finally, define $\bar{\pi}^\circ$ by $\bar{x}_i^\circ=\bar{x}_{m+i-1}$ for $i=1,2,\ldots,1+q$ and $\bar{x}_i^\circ=0$ for $i=2+q,\ldots,p$. Then

$$\begin{aligned}\sigma(\bar{\pi}^\circ) &= \bar{x}_1^\circ\cdots\bar{x}_{1+q}^\circ \\ &= \bar{x}_m\cdots\bar{x}_n \\ &= \sigma(\bar{\pi}) \\ &= \mu. \blacksquare\end{aligned}$$

Henceforth $\bar{\pi}$ denotes the partition of theorem 2.3.

Theorem 2.4 In $\bar{\pi}$, $\bar{x}_1=\ldots=\bar{x}_{1+q}$.

Proof. Let $\pi=(\underbrace{\frac{\alpha}{q+1},\ldots,\frac{\alpha}{q+1}}_{q+1},0,\ldots,0)$. Then $\mu\geq\sigma(\pi)>0$.

Hence in $\bar{\pi}$ it must be true that $\bar{x}_i>0$ for $i=1,\ldots,1+q$.

Let π be such that $x_i=0$ for $i=2+q,\ldots,p$, $x_i>0$ for $i=1,\ldots,1+q$, but that $x_h\neq x_k$ for some h and k in the range $1\leq j,k\leq 1+q$. We show $\mu\neq\sigma(\pi)$.

Assume without loss of generality that $x_h>x_k$. Select ε to be $\varepsilon=\frac{x_h-x_k}{2}$.

Then $\sigma(\pi<k,h>) = y_1 \cdots y_{1+q}$

$$= \frac{(x_h-\varepsilon)(x_k+\varepsilon)}{x_h x_k} x_1 \cdots x_{1+q}$$

$$= x_1 \cdots x_{1+q} + \frac{\varepsilon x_h - \varepsilon x_k - \varepsilon^2}{x_h x_k} x_1 \cdots x_{1+q}$$

$$= \sigma(\pi) + \varepsilon^2 \cdot \frac{x_1 \cdots x_{1+q}}{x_h x_k}$$

$$> \sigma(\pi).$$

Thus $\mu \neq \sigma(\pi)$, and the theorem is established. ∎

Theorem 2.5 For $\bar{\pi} = (\underbrace{\frac{\alpha}{1+q}, \ldots, \frac{\alpha}{1+q}}_{1+q}, 0, \ldots, 0)$, $\mu = \sigma(\bar{\pi}) = (\frac{\alpha}{1+q})^{1+q}$.

Proof. The proof is immediate from theorem 2.4. ∎